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STRENGTH OF MATERIALS

BY

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THIRD EDITION
NINTH IMPRESSION

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PREFACE TO THE THIRD EDITION

The reader's attention is called to several changes and additions that have been made in this third edition of "Strength of Materials." On account of the great increase in the use of welding, two articles on welded joints have been added. Torsion of shafts has been treated earlier in the text, and an article on shearing stresses in noncircular shafts has been added. All the material on elastic resilience has been placed in one chapter. The subjects of shear centers of channel sections, H-sections with unequal flanges, and angle sections have been added. Continuous beams with fixed ends and unequal loads or spans have been treated. Moments and stresses in beams due to moving loads have been discussed.

In this edition, the deflection of beams has been treated in only two ways, by the double integration method and by the area moment method. The chapter on the equivalent cantilever method has been omitted, since it is felt that it really belongs in a course in advanced strength of materials.

In places in the text in which the author's experience with his students appeared to demand it, additional illustrative examples have been solved. Except in a very few cases where certain typical problems are necessary, entirely new problems have been used. As in the former edition, the answers to all problems have been given.

As in the second edition, the author wishes to acknowledge his indebtedness to Director Newlin and to the Forest Service of the U. S. Department of Agriculture for the valuable tables on timber, and to the American Institute of Steel Construction and the Building Code Committee of the U. S. Department of Commerce for permission to reprint the tables of specifications for structural steel.

A. P. Poorman.

Purdue University,
March, 1937.

PREFACE TO THE FIRST EDITION

This book on Strength of Materials is intended primarily for use in undergraduate courses in Mechanics, and is a companion volume to the author's book "Applied Mechanics." A working knowledge of the principles of physics, the calculus, and statics is assumed. The theoretical elastic behavior of engineering materials is treated concurrently with the results of experiment.

In this book, as in "Applied Mechanics," a large number of illustrative examples have been worked out in detail to aid the student in mastering the relation between theory and application. The problems at the end of each article form an integral part of the treatment of the subject and should be mastered thoroughly. The answers to all problems have been given for the encouragement of the student. Teachers who prefer that no answers be given may accomplish that result by making proper changes in the data of the problems.

In the discussion of reinforced concrete beams, the "equivalent area" method has been used. A trial of this method for several years in the author's classes has convinced him that his treatment has decided advantages over the one commonly used.

The deflection of beams has been treated in three different ways. In Chapter VII, the widely used double integration method is given. In Chapter XVIII, the area moment method is developed and its application to some of the simpler cases is shown. In Chapter XIX, the equivalent cantilever method is developed. In the simpler cases, this method has no advantages over either of the other methods, but in the more complex cases its advantages become very evident. For the principles underlying this method, the author is indebted to his colleague, Professor C. H. Lawrance. In an excellent article on the subject, Professor Lawrance has developed this method much more fully than the limits of this book will permit, and has shown its application to the theorem of three moments, to continuous beams with fixed ends, to portals of bridges, and to bents of steel mill buildings.

The student should provide himself with a standard steel handbook, even though some of the tables of properties of rolled steel shapes have been placed at the end of this book for handy reference. For permission to reprint Tables IX to XII from the Carnegie Pocket Companion, the author wishes to express his thanks to the Carnegie Steel Company of Pittsburgh, Pennsylvania.

The author wishes to acknowledge his indebtedness to the standard books on Mechanics, in particular to the classic works of Rankine and Morley which have been used freely for reference, and to those of Merriman, Church and Boyd, which have been used by the author as text books in his classes.

A. P. POORMAN.

PURDUE UNIVERSITY,
July, 1925.

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STRENGTH OF MATERIALS

CHAPTER I

STRESSES AND STRAINS IN TENSION AND COMPRESSION

1. Definitions.—*Strength of materials* is that branch of mechanics which treats of the forces acting upon elastic bodies and the resulting deformations. The elastic bodies which are considered are such as those which compose machines and structures used in engineering work.

Stress is the internal force exerted by one part of a body upon the adjoining part.

Strain or deformation is the change in the dimensions of a body caused by stress.

According to Hooke's law, a perfectly elastic body is one in which the strain in the direction of the stress is always proportional to the stress. No materials are perfectly elastic, but it has been found by experiment that many materials used in engineering approximate this condition closely within certain limits. Since varying conditions, such as temperature, heat treatment, and previous mechanical treatment, change markedly the degree of elasticity of any given substance, results of experiments must continually be used to determine how much or how little is the variation of the behavior of any given material from the theoretic behavior of the perfectly elastic solid.

A *bar* is commonly defined as a solid of prismatic form, that is, one having the same size throughout its length, as shown in Fig. 1(a). If a plane which intersects a bar is normal to the axis of the bar, the intersection of the plane and the bar is called the *cross section* of the bar. The area of this intersection is called the *section area,* as shown at A, in Fig. 1(a).

An *axial force* is one which acts along the geometric axis of a bar, as force P in Fig. 1(a). If the resultant of a system of distributed forces applied to a bar acts along its axis, the system is equivalent to an axial force.

2. Engineering Materials and Their Mechanical Properties.—Before beginning a theoretical discussion of the relations between stresses and deformations of elastic solids, it will be well to consider briefly some of the materials in common use in modern engineering work and their mechanical properties. The chief of these are timber, stone, brick, concrete, cast iron, wrought iron, steel, copper, and aluminum.

Timber is light, elastic, resilient, and nonductile. As shown in Table I at the end of this volume, the average weight of timber is about 36 pounds per cubic foot. Table XIII gives the average weights of twenty-four species of American hard or deciduous timber, as determined by the Forest Products Laboratory, U. S. Forest Service, Madison, Wisconsin. These weights vary from 26 to 62 pounds per cubic foot at 12 per cent moisture content. The average weight of the twenty-four species is 42 pounds per cubic foot.

Table XIV gives the average weights of twenty-four species of American soft or conifer timber, which vary from 22 to 41 pounds per cubic foot and average 29.7 pounds per cubic foot. The average of the entire forty-eight species is 36 pounds per cubic foot.

On account of the fibrous and cellular structure of timber, its strength and elastic properties parallel to the grain and perpendicular to the grain are entirely different.

Stone is low in elasticity and resilience, nonductile, high in compressive strength, and low in tensile strength. Stone usually has either sedimentary or pressure cleavage planes, in which case its strength is not the same in all directions. Compressive stresses should always be applied to stone normal to the cleavage planes. Tensile and flexural stresses should always act parallel to the cleavage planes. The average weight of stone is about 160 pounds per cubic foot.

Brick is made from clay or shale which has been dried, pulverized, moistened into mud, molded into proper shape, and burned. Brick is lighter than stone, its weight being only 125 pounds per cubic foot, but, if well burned, its strength approaches that of stone. The other properties of brick are very much the same as those of stone.

Concrete is composed of an aggregate of stone, or of sand and stone, bound together by hydraulic cement. On account of the materials of which it is composed, the properties of concrete are very similar to those of stone. Its tensile strength is exceptionally low. The average weight of concrete is about 150 pounds per cubic foot.

Cast iron is a metal produced from iron ores by a process of reduction in a blast furnace and refining in a cupola furnace. Cast iron is very high in compressive strength, but is low in tensile strength, ductility, and malleability. A fractured surface shows a crystalline structure. Small iron castings may be made malleable and increased in tensile strength by annealing in contact with decarbonizing material. The addition of nickel increases the tensile strength enormously. An amount as small as 2 per cent will more than double the tensile strength. The average weight of cast iron is about 450 pounds per cubic foot.

Wrought iron is the term applied to iron which has been purified by melting in a reverberatory puddling furnace and by subsequent working while in a pasty condition. Wrought iron is comparatively soft and ductile, very tough, malleable, and easily welded. It is heavier than cast iron, weighing about 480 pounds per cubic foot. In the process of working, the impurities it contains in the form of slag are drawn out in such a way as to give the metal a fibrous structure, particularly noticeable when fractured or when etched by acid. The strength of wrought iron is about the same in tension and compression. Wrought iron has been largely replaced by steel, especially in structural work, but on account of its superior resistance to corrosion, it is coming back into use, especially in sheet-metal work.

Steel, in its softer grades, closely resembles wrought iron both in chemical composition and in physical properties. The impurities contained in the molten iron are removed by a process which involves complete fusion, and as a result the structure is granular instead of fibrous. Steel is a homogeneous material and has practically the same strength in tension and compression. It is malleable, ductile, easily welded, and in the harder grades may be tempered. Its weight is about 490 pounds per cubic foot. Ordinary steel is subject to corrosion, a characteristic which tends to neutralize its cheapness. The addition to steel of small amounts of nickel, tungsten, vanadium, chromium, or molybdenum adds enormously to the strength, hardness, and toughness of the metal. Such steels are called alloy steels and have a wide use in modern engineering. The addition of 8 per cent nickel and 18 per cent chromium to steel makes the familiar stainless steel, which is very resistant to corrosion under nearly all conditions.

Copper is a reddish metal, highly malleable and ductile, and is an excellent conductor of electricity. It is also highly resistant to corrosion, either alone or in alloys with other metals. An amount of copper as low as 0.2 per cent improves markedly the resistance of steel to corrosion. Copper is alloyed with zinc to form brass and with tin to form bronze. An alloy of copper with nickel called monel metal is extracted from an ore containing both metals without separating them. Copper weighs about 556 pounds per cubic foot.

Aluminum is a light, white metal, with a fair degree of ductility and malleability. It is highly resistant to corrosion and has an electrical conductivity about 60 per cent that of copper. It weighs about 165 pounds per cubic foot. Its strength is very low, about one-third that of steel, but if it is alloyed with about 3 per cent of copper to form *duralumin*, its strength may be made to approach that of steel, while its weight is increased only slightly.

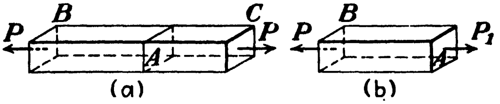


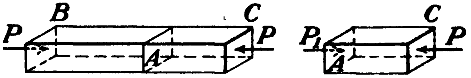
FIG. 1.

In addition to these principal engineering materials, there are many secondary materials, valuable on account of some special property. Among these may be mentioned Dow metal, a magnesium alloy weighing only about 115 pounds per cubic foot; rubber, valuable for its extreme range of elasticity; a synthetic rubber, called duprene, highly resistant to action by oils; glass, valuable for its transparency; and cordage and leather, valuable for their lightness, toughness, and flexibility.

3. Stresses in Tension and Compression.—Figure 1(*a*) represents a bar *BC* which is acted upon by two equal and opposite axial forces *P*, *P*. These forces tend to pull the bar apart and are called *tensile* forces. The bar is said to be in tension. If, now, the part of the bar on the left of section *A* is considered as a "free body," there must be placed upon it at the section *A*, a force P_1 which represents the action of the part removed upon the part remaining, as shown in Fig. 1(*b*). The force P_1 is the resultant of the system of distributed forces which compose the internal stress at the section. These internal forces are considered to be uniformly distributed over the cross section, so their resultant is axial. It is necessary, also, that force P_1 be equal and opposite to force P, since it must hold force P in equilibrium.

In a similar manner Fig. 2(*a*) represents a bar BC in compression, since it is acted upon by two equal and opposite axial forces P, P directed toward each other. In Fig. 2(*b*), the part of the bar on the right of section A is shown as a free body, upon which the force P_1, equal and opposite to force P, replaces the action of the left part which has been removed.

The external compressive forces P, P at the ends of the bar are also called *bearing stresses.*



(a) (b)

FIG. 2.

The *unit stress* at any section in a bar is the stress upon a unit area of the section, and its average value is obtained by dividing the total axial load P by the area of the cross section A. If S represents the average unit stress,

$$S = \frac{P}{A}.$$

The unit of area most commonly used by engineers in the United States is the square inch, and stresses are expressed in units of pounds per square inch, abbreviated, "lb./in.^{2}." The bearing pressure of masonry upon its supporting soil or foundation is commonly expressed in units of tons per square foot. British engineers express stresses in units of long tons (2240 pounds) per square inch, and Continental engineers use kilograms per square centimeter.

If a vertical bar rests upon a horizontal base and carries no load but its own weight, the total stress at any horizontal cross section is the weight of the part of the bar above that cross section. The unit stress at any cross section is obtained by dividing the weight of the part of the bar above that cross section by the area of the cross section.

The unit stress at any section is also given by the weight of a bar which has unit cross section and a length equal to the height above that cross section.

If a bar is suspended from its upper end and carries no load but its own weight, the same method of solution is used.

PROBLEMS

1. A tie rod 2.5 in. in diameter supports a load of 75,000 lb. Compute the unit stress S. *Ans.* 15,280 lb./in.^{2}.

2. A short timber block 9.6 by 15.8 in. supports a load of 182,000 lb. by means of a steel bearing block. Compute the unit stress S in the timber block. *Ans.* 1200 lb./in.^{2}.

3. Compute the diameter of a bar to hold a load of 800 lb. with a stress of 22,000 lb./in.^{2}. *Ans.* 0.215 in.

4. Compute the load necessary to cause a stress of 1000 lb./in.^{2} in a bar 1.08 in. in diameter. *Ans.* 916 lb.

5. A short piece of steel pipe 12.75 in. outside diameter and 12.00 in. inside diameter supports an axial load of 250,000 lb. Compute the unit stress. *Ans.* 17,150 lb./in.^{2}.

6. Compute the height of a concrete chimney to cause a stress of 200 lb./in.^{2} direct compressive stress at the base, assuming concrete weighs 150 lb./cu. ft. *Ans.* 192 ft.

7. If the concrete chimney referred to in Problem 6 is 12 ft. outside diameter and 8.5 ft. inside diameter, what must be the size of a square concrete base 2 ft. thick to support it if the allowable bearing pressure on the supporting soil is 2 tons/ft.^{2}? *Ans.* 20.94 ft.

8. A steel cable 800 ft. long composed of nine wires each 0.148 in. in diameter, is hung vertically in a mine shaft and carries a load of 2500 lb. Compute the stress at the upper end due to its weight alone, and that due to the load alone. Steel weighs 490 lb./cu. ft. Neglect the angularity of the wires. *Ans.* 2720 lb./in.^{2}; 16,150 lb./in.^{2}.

4. Strains or Deformations in Tension and Compression.—A bar which is subjected either to a tensile force or to a compressive force will change in length, being lengthened by the tensile force and shortened by the compressive force. A bar of soft rubber illustrates this property very well. It will be found that when such a bar has a small load applied in tension, it will be elongated an amount which may easily be measured, while its cross section is decreased. If another load of the same amount is added, the total elongation will be twice as much, in accordance with Hooke's law. If a third load of equal amount is added, there will be another equal increase in the elongation, making the total elongation three times what it was for one load. If, now, the loads are all removed, the bar will return to its original length and cross section.

In a similar manner, successive equal loads in compression will shorten the bar equal amounts and will increase its cross section. When the loads are removed, the bar regains its original size and shape.

Practically all other engineering materials behave similarly, except that the amount of the deformation is extremely small. While a rubber bar may be stretched to two or three times its original length without apparent injury, a structural steel bar, for instance, may be stretched only about 0.0015 of its original length without injury.

The unit strain in a bar is the deformation in a unit length of the bar. If both the load and the cross section of the bar are constant, the unit strain is obtained by dividing the total deformation by the original length. The Greek letter δ (delta) will be used to represent the unit strain. If l is the original length of the bar and e the total deformation,

$$\delta = \frac{e}{l}.$$

If either the load or the cross section varies, the quantity e/l gives only the average unit strain. The exact value of the unit strain at any certain section of the bar is given by the differential expression, $\delta = de/dl$.

If a bar is suspended from its upper end and carries only its own weight, the unit strain varies uniformly from zero at the lower end to a maximum value at the upper end. The average unit strain is given by the expression $\delta = e/l$, but the maximum unit strain at the upper end is $\delta_1 = 2e/l$.

5. Elastic Limit.—As stated in Art. 4, it is found that for each increment in the stress on a bar there is a closely proportional increase in the strain, provided a certain limit of stress is not exceeded. If a stress greater than this limiting stress is applied, the corresponding strain is not proportional to the stress, but increases more rapidly. The unit stress at this limit is called the *proportional elastic limit,* and is defined concisely as follows:

The proportional elastic limit is the maximum unit stress up to which the strain is proportional to the stress.

It is found, also, that if a stress is applied to a bar and then removed, the resulting deformation or strain disappears, provided a certain limit of stress has not been exceeded. If a stress greater than this limit is applied, the resulting strain will not all disappear when the stress is removed, but a part of it will remain as a *permanent set.* The unit stress at this limit is also called the elastic limit, and for nearly all materials agrees very closely with the proportional elastic limit. By this definition, *the elastic limit is the maximum unit stress which may be applied to a bar without producing a permanent set.*

An accurate determination of the elastic limit, as just defined, evidently depends upon the detection of the smallest amount of permanent set. Tests with very precise instruments and with small increments of load will therefore give lower values than tests with less precise instruments and larger increments of load. Rapidity of loading and duration of loading affect the values obtained to a marked degree. The transmission and distribution of the applied load and its balancing by the induced internal stresses are not instantaneous. On this account, higher values of the elastic limit will be obtained if the loading is rapid than if it is slow. Higher values will also be obtained if the readings of the deformations are made as soon as the loads are applied, than will be obtained if some appreciable time is permitted to elapse.

Theoretically there is a permanent set for any load, however small, since no material solids are perfectly elastic, but with ordinary instruments as commonly used, no permanent set is noticed until after the proportional elastic limit has been exceeded. Unless otherwise stated, the term *elastic limit* as used in this text will mean the *proportional elastic limit.* Table II gives average values of the elastic limit for some common engineering materials. Tables XIII and XIV give the values of the elastic limit of a number of species of timber as determined by tests at the Forest Products Laboratory, U. S. Forest Service, Madison, Wisconsin.

PROBLEMS

1. Compute the length of a vertical aluminum rod suspended at the upper end in which the stress at the point of suspension just reaches the elastic limit. *Ans.* 10,470 ft.

2. Compute the length of a copper wire suspended in water in which the stress at the upper end is one-half of the elastic limit. *Ans.* 1460 ft.

3. In a test of an 8-in. concrete cylinder in compression, six readings between zero and the elastic limit are desired. Compute the necessary increment of load. *Ans.* 8377 lb.

4. A steel wire 1200 ft. long and 0.18 in. in diameter for which the elastic limit is 80,000 lb./in.^{2}, is suspended in a mine shaft. What load in addition to its own weight will cause the stress at the upper end to reach the elastic limit? *Ans.* 1932 lb.

6. Modulus of Elasticity.—*Young's modulus* or the *modulus of elasticity* is the ratio of the unit stress to the corresponding unit strain for any value of the unit stress below the proportional elastic limit. As stated in Art. 4, this ratio is practically constant, and may therefore be obtained by dividing any increment of unit stress by the corresponding increment of unit strain. The same relation holds true and practically the same values are obtained in both tension and compression for the common engineering materials.

The modulus of elasticity will be represented by E and will be used in units of pounds per square inch. If S represents the unit stress and δ the corresponding unit strain,

$$E = \frac{S}{\delta}.$$

It will be seen that if δ becomes unity, $E = S$, so the modulus of elasticity may be considered as the unit stress which would stretch a bar to twice its original length, provided the elastic laws remained true through that range. Rubber is the only material for which the elastic laws hold true for this amount of deformation. As stated in Art. 4, the elastic range for structural steel is only about 0.0015 of this amount.

Table II gives average values of the modulus of elasticity for some common engineering materials. Tables XIII and XIV give the values of the modulus of elasticity of a number of species of timber as determined by tests at the Forest Products Laboratory, U. S. Forest Service, Madison, Wisconsin.

PROBLEMS

1. A steel bar 0.50 in. in diameter when tested in tension, stretched an amount of 0.0069 in. in a length of 8 in. with a load of 5000 lb. Compute S, δ, and E. *Ans.* 25,460 lb./in.^{2}; 0.0008625; 29,500,000 lb./in.^{2}.

2. A concrete cylinder 8 in. in diameter tested in compression, shortened an amount of 0.0032 in. in a length of 15 in. with a load of 20,000 lb. Compute E. *Ans.* 1,870,000 lb./in.^{2}.

3. A deep-well pump rod is 186 ft. long. If it is made of monel metal weighing 553 lb. per cu. ft. and $E = 15{,}000{,}000$ lb./in.^{2}, how much is it stretched by its own weight? If it is ½ in. in diameter, how much is it stretched when under a tension of 500 lb.? *Ans.* 0.0531 in.; 0.379 in.

4. Compute the pull necessary to stretch a steel wire 0.2 in. in diameter and 750 ft. long an amount of 2 in. if $E = 26{,}000{,}000$ lb./in.^{2}.
Ans. 182 lb.

7. Stresses in Bars of Two Materials.—If a bar carrying tensile or compressive loads is composed of two materials with different moduli of elasticity, the material with the lower modulus will take less unit stress than the other for the same deformation. Let δ be the unit strain, S_1 and S_2 the unit stresses, E_1 and E_2 the moduli of elasticity, and P the total load applied. Then, since $\delta = S/E$,

$$\frac{S_1}{E_1} = \frac{S_2}{E_2},$$

or

$$S_2 = \frac{E_2}{E_1}S_1.$$

Let the ratio of the moduli, E_2/E_1, be called n_2. Then

$$S_2 = n_2S_1,$$

and

$$P = A_1S_1 + A_2S_2 = (A_1 + n_2A_2)S_1.$$

The quantity n_2A_2 may be called the *equivalent area*, and is the amount of area of the one material which would replace the area A_2 of the other, and have the same unit strain.

If there are three different materials, a similar procedure is followed. Areas A_2 and A_3 are replaced by their equivalent areas, n_2A_2 and n_3A_3.

EXAMPLE

A copper bar 2 in. wide and ½ in. thick is placed between two structural steel bars each 2 in. wide and ⅜ in. thick. Compute the unit stress and the unit strain in each material caused by a tensile load of 20,000 lb. Compute the elongation in a length of 2 ft.

Solution.—Let A_1 be the area of the copper and A_2 the area of the steel. $A_1 = 1$ sq. in., $A_2 = 1.5$ sq. in., $E_1 = 15{,}000{,}000$ lb./in.^{2}, and $E_2 = 29{,}000{,}000$ lb./in.^{2}.

$$n_2 = \frac{E_2}{E_1} = \frac{29}{15} = 1.933.$$

$$n_2A_2 = 1.933 \times 1.5 = 2.9 \text{ sq. in.}$$

This area, 2.9 sq. in., is the area of copper which would be equivalent to the 1.5 sq. in. of steel. The original area of the copper plus the copper equivalent of the area of the steel is then

$$A_1 + n_2A_2 = 1 + 2.9 = 3.9 \text{ sq. in.}$$
$$S_1 = \frac{20{,}000}{3.9} = 5128 \text{ lb./in.}^2.$$
$$S_2 = n_2S_1 = 1.933 \times 5128 = 9914 \text{ lb./in.}^2.$$
$$\delta = \frac{5128}{15{,}000{,}000} = 0.000342.$$
$$e = l\delta = 24 \times 0.000342 = 0.0082 \text{ in.}$$

PROBLEMS

1. A concrete cylinder 8 in. in diameter and 16 in. long is reinforced with four structural steel rods, each 1 in. in diameter. Compute the unit stress in each material and the total deformation due to an axial compressive load of 80,000 lb.

Ans. $S_1 = 864$ lb./in.²; $S_2 = 12{,}500$ lb./in.²; $e = 0.006912$ in.

2. A block of timber 16 in. square is reinforced with a 6- by 6- by 1-in. structural steel angle at each corner. Using the value of E_w as 1,750,000 lb./in.², compute the total axial load which will cause a stress of 18,000 lb./in.² in the steel. Compute the unit stress in the timber.

Ans. 1,070,000 lb.; 1087 lb./in.².

3. A piece of structural steel pipe 12.5 in. outside diameter and 12 in. inside diameter is filled with concrete. If the allowable stress in the steel is 18,000 lb./in.² and that in the concrete is 1000 lb./in.², what total axial load may be applied? *Ans.* 252,600 lb.

4. A 1-in. steel bolt extends through a brass tube which is 1 in. inside diameter, 1.6 in. outside diameter, and 12 in. long. The threads on the bolt have a pitch of 0.15 in. Compute the unit stress in each material when the nut on the bolt is turned down one-tenth of a turn.

Ans. $S_1 = 15{,}570$ lb./in.²; $S_2 = 9980$ lb./in.².

5. A cast-iron cylinder 4 in. in diameter is surrounded by a concrete block 12 in. square, at each corner of which is a structural steel angle 6 by 6 by ½ in. Compute the unit stress in each material due to an axial load of 600,000 lb. *Ans.* 8050 lb./in.²; 1073 lb./in.²; 15,560 lb./in.².

8. Temperature Stresses.—Through ordinary ranges of temperature, bodies expand as their temperature rises, and contract as their temperature falls. The relative change of dimension per degree of temperature is called the *coefficient of temperature* and is here represented by the Greek letter ϵ (epsilon). Table VI gives values of the coefficients of temperature for several materials for both Fahrenheit and centigrade scales.

If a bar is free to expand or contract as its temperature rises or falls, there will be no change in its condition of internal stress. If, however, a bar is partly or wholly restrained, the same internal stress will be induced, below the elastic limit, as would be necessary to cause the same change in length. As shown in Art. 6, $S = E\delta$. If t is the number of degrees change in temperature, $\delta = \epsilon t$, so $S = E\epsilon t$.

PROBLEMS

1. A steel fence wire 0.148 in. in diameter is stretched between fixed end posts with a tension of 300 lb. when the temperature is 90°F. If the elastic limit of the wire is 40,000 lb./in.^{2}, to what temperature could the wire fall without causing a permanent set? *Ans.* −29.7°F.

2. Steel railroad rails 39 ft. long are laid in a track with spaces of $\frac{1}{8}$ in. between their ends when the temperature is 40°F. What will be the space between them when the temperature is −20°F.? At what temperature will the rails just touch? What will be the unit stress in the rails when the temperature is 140°F.? *Ans.* 0.3075 in.; 81°F.; 11,100 lb./in.^{2}.

3. The inside diameter of a locomotive tire is 71.97 in. at a temperature of 60°F. Compute the temperature when the diameter is 72.06 in. If the tire is then placed on the wheel and the inside diameter of the tire is 72.00 in. when it is cooled again to 60°F., what is the unit stress in the tire? *Ans.* 252.6°F.; 12,100 lb./in.^{2}.

9. Poisson's Ratio.—Experiment has shown that when a bar is subjected to a tensile stress, its transverse dimensions are shortened at the same time that its axial dimension is lengthened. Similarly, when a bar is subjected to a compressive stress, its transverse dimensions are lengthened at the same time that its axial dimension is shortened. Below the elastic limit, the transverse deformation is proportional to the stress the same as is the axial deformation. Since both the unit lateral deformation and the unit axial deformation are proportional to the stress, their ratio is constant. This ratio of the unit lateral deformation to the unit axial deformation is called *Poisson's ratio,* and is denoted by m. The average value of m for cast iron is about 0.25, the value for wrought iron and steel is about 0.30, and for brass and copper it is about 0.33.

In case direct axial stresses are applied in two directions or in all three directions, each axial stress produces its deformation independently of the others, and the total change in any one dimension is the algebraic sum of all of the deformations in that direction.

Since stresses are proportional to deformations, the equivalent stress in any direction is given by the expression $S = E\delta$.

EXAMPLE 1

The steel bar shown in Fig. 3 is 2 in. wide, 1 in. thick, and 12 in. long. If the value of E is 30,000,000 lb./in.², compute the new dimensions of the bar and the equivalent lateral unit stress due to an axial tensile load of 60,000 lb.

Solution.—Let the X, Y, and Z axes be placed as shown in Fig. 3. The direct unit stress in the X direction is S_x = 60,000/2 = 30,000 lb./in.². The unit axial elongation due to this load is $\delta_x = S/E$ = 0.001 in., and the total elongation is e_x = 12 × 0.001 = 0.012 in. Each lateral dimension is shortened, the unit shortening being 0.3 × 0.001 = 0.0003 in. The 2-in. dimension then becomes 2 − 2 × 0.0003 = 1.9994 in. The 1-in. dimension becomes 1 − 0.0003 = 0.9997 in., and the 12-in. dimension becomes 12 + 0.012 = 12.012 in. The equivalent lateral unit stress in each direction is 0.0003 × 30,000,000 = 9000 lb./in.².

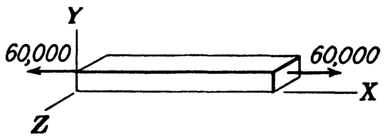


FIG. 3.

EXAMPLE 2

If the bar described in Example 1 has an additional load of 648,000 lb. in compression parallel to the Y axis, compute the new dimensions and the equivalent stress in each direction.

Solution.—Parallel to the Y axis, the direct unit stress is S_y = 648,000/24 = 27,000 lb./in.². The 1-in. dimension is shortened an additional amount of δ_y = 27,000/30,000,000 = 0.0009 in., and so becomes 1 − 0.0003 − 0.0009 = 0.9988 in. The 2-in. dimension is lengthened by an amount of 2 × 0.3 × 0.0009 = 0.00054 in. and so becomes 2 − 0.0006 + 0.00054 = 1.99994 in. The 12-in. dimension is lengthened an additional amount of 12 × 0.3 × 0.0009 = 0.00324 in., and becomes 12 + 0.012 + 0.00324 = 12.01524 in.

The unit elongation in the 12-in. dimension is 0.01524/12 = 0.00127 in., and the equivalent unit stress is 0.00127 × 30,000,000 = 38,100 lb./in.² tension. The unit shortening in the 2-in. dimension is 0.00006/2 = 0.00003 in., and the equivalent unit stress is 0.00003 × 30,000,000 = 900 lb./in.² compression. The unit shortening in the 1-in. dimension is 0.0012 in., and the equivalent unit stress is 0.0012 × 30,000,000 = 36,000 lb./in.² compression.

PROBLEMS

1. If the loads mentioned in the examples above are removed, and a load of 288,000 lb. in compression is applied parallel to the Z axis, what are the new dimensions of the bar? *Ans.* 12.00288 in.; 1.00024 in.; 1.99840 in.

2. If the load mentioned in Problem 1 is in addition to the two mentioned in the examples, what are the equivalent stresses in the three directions?

Ans. 45,300 lb./in.² tension; 28,800 lb./in.² compression; 24,900 lb./in.² compression.

3. A steel reinforcing bar is 0.8760 in. in diameter when unstressed. Compute its diameter when stressed to 18,000 lb./in.^{2} in tension if $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 0.87584 in.

4. A steel punch is 1.0625 in. in diameter and 3.380 in. long. What are its dimensions when it is exerting a pressure of 130,000 lb. to punch a hole through a plate? $E = 30{,}000{,}000$ lb./in.^{2}. *Ans.* 1.06406 in.; 3.3635 in.

10. Stress-strain Diagrams.—The relation between the stress applied to a bar and the resulting strain or deformation is shown graphically by means of the *stress-strain* diagram. In making tests from which to obtain data for plotting stress-strain diagrams, it is necessary to take simultaneous readings of the load and the deformation, from which the values of the unit stresses and the corresponding unit strains may be computed. The usual method of making such tests is to apply a certain load to the test piece by means of a testing machine. In the usual form of testing machine the amount of the load applied to the test piece is measured by moving a counterweight along a graduated scale beam until it balances the pressure exerted through the test piece upon the weighing platform. The value of the deformation is determined by the readings of an instrument which measures accurately the distance between two given cross sections of the test piece.

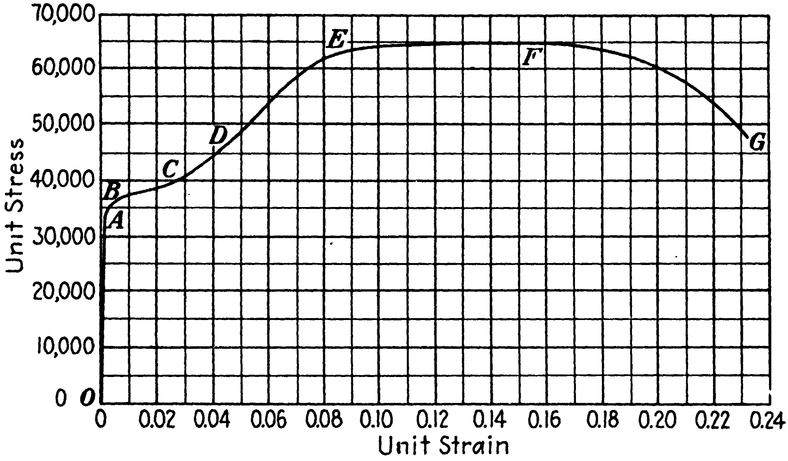


FIG. 4.—Stress-strain diagram of medium steel.

The stress-strain diagram is then drawn by plotting a series of points whose ordinates represent different values of the unit stress to which the test piece was subjected, and whose abscissas represent corresponding values of the unit deformation. Through these points a line is drawn which is called the ***stress-strain curve.***

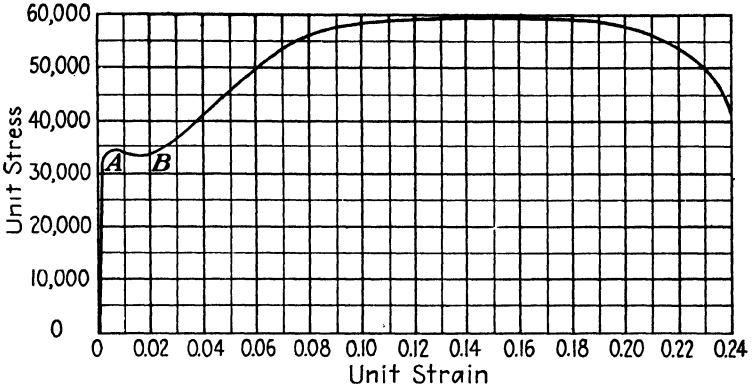


FIG. 5.—Stress-strain diagram of boiler-plate steel.

The diagrams for medium structural steel, boiler-plate steel, and wrought iron resemble each other very closely, as shown in Figs. 4, 5, and 6. The diagrams shown are for tests in tension, but tests in compression give similar curves.

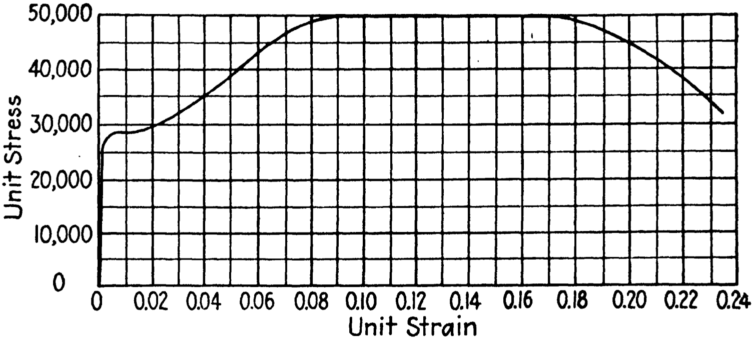


FIG. 6.—Stress-strain diagram of wrought iron.

In tests of cast iron, the curves show a great difference in tension and compression, as shown in Fig. 7. The elastic limit in compression is more than twice as great as in tension, and the ultimate strength in compression is about four times as great as in tension.

The curves for timber in tension and in compression parallel to the grain (Fig. 8) are very close together except for the values of the ultimate strength. The ultimate strength in tension is appreciably higher.

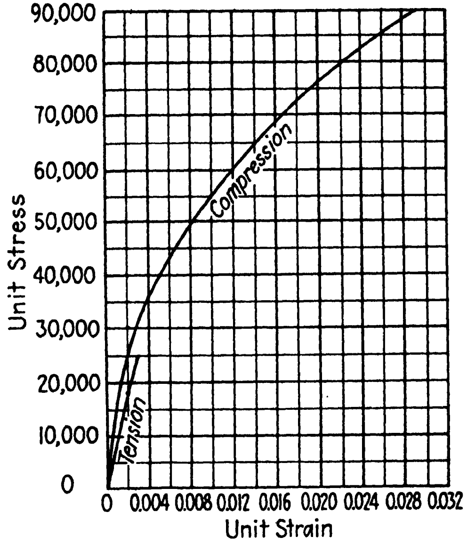


FIG. 7.—Stress-strain diagram of cast iron.

In the diagram for concrete (Fig. 9) the curve for compression is the only one shown. Accurate tests of concrete in tension are very difficult to make and are of small practical value, since concrete should not be used in tension.

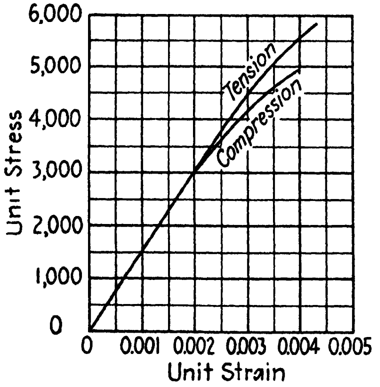


FIG. 8.—Stress-strain diagram of timber.

In Fig. 10 are shown the curves for brass and copper, both in tension and in compression. These curves are typical of ductile materials. Such materials do not reach an ultimate value in compression, but merely continue to increase indefinitely in cross-section area as the load increases.

Since the modulus of elasticity E is the ratio of any increment of unit stress to the corresponding increment of unit strain up to the elastic limit, the numerical value of E is given by the slope of the straight part of the stress-strain curve in terms of the units used in plotting the diagram.

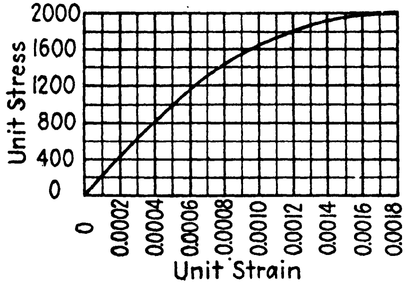


FIG. 9.—Stress-strain diagram of concrete.

11. Yield Point, Ultimate Strength, and Rupture Strength.—In Fig. 4 is shown the stress-strain diagram for medium structural steel in tension, a diagram which is typical of ductile metals. Up to point A on the curve the strain is proportional to the stress, so the curve from O to A is a straight line. The unit stress at A where this proportionality ceases is the *proportional elastic limit*, which for steel and wrought iron agrees closely with the elastic limit up to which a bar may be stressed with no permanent set. In commercial tests of steel and wrought iron in tension, no attempt is made to determine the elastic limit, as no measurements of the deformation are made.

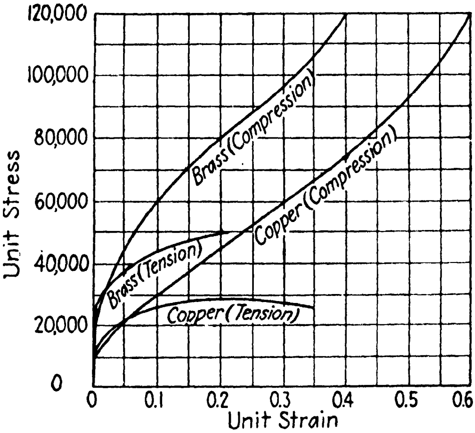


FIG. 10.—Stress-strain diagram of brass and copper.

For stresses above the elastic limit, the strain increases at a faster rate, until at point B on the curve the deformation continues with little or no increase in the load. This unit stress is called the *yield point.* Bars which have been rolled hot and have not been machined are coated with an oxidized mill scale which is not elastic. As the metal begins to yield, the inelastic mill scale begins to fall off in small flakes. Since the material is neither absolutely homogeneous nor uniform in cross section, all of the test piece does not reach the yield point at the same time. At one or more places the metal will begin to stretch and the mill scale to fall, while the remainder of the bar apparently remains unchanged. The area where the mill scale is falling gradually spreads until the entire bar is stretched, during which time the load remains constant or decreases slightly. The harder grades of steel and other nonductile metals have no true yield point, but show only a slight bend in the stress-strain curve just above the elastic limit.

After all of the bar has passed the yield point, it begins to recover strength, so that in order to continue the elongation it is necessary to increase the load, slowly at first, as at C, Fig. 4, then more rapidly as at D, then more slowly again as at E. For a second time the increase in load slows down until finally the stretching continues with no further increase in the load, as at F. This load is the *maximum load* and the unit stress at this load is called the *ultimate strength* of the material.

After the load has remained constant a short time at this maximum value, the additional stretch becomes localized and a neck is formed. The load then decreases and finally the bar breaks at a load somewhat less than the maximum load. This is the *rupture load,* and the unit stress at rupture is called the *rupture strength* of the material. Materials which do not form a neck when tested in tension have a continually rising curve, and the ultimate strength and the rupture strength are the same.

All three of these stresses, the yield point, the ultimate strength, and the rupture strength, are the *nominal* stresses, that is, they are computed by using the original area of the cross section.

In tests in compression, ductile materials have no true ultimate strength. Except for the frictional resistance to expansion at the bearing surfaces, such test pieces deform plastically after the elastic limit is reached. The volume remains constant, so the cross-section area increases as the length decreases, and the bar continues to change in shape without fracture. Hard and brittle materials develop stresses and strains in compression very much as they do in tension, and finally fail by splitting or shearing along some diagonal plane or planes.

Table II at the end of this volume gives the average values of the ultimate strength of a number of engineering materials.

PROBLEMS

1. A steel bar 0.874 in. in diameter held a load of 22,100 lb. in tension when the mill scale began to fall, a load of 38,600 lb. at the maximum, and a load of 33,000 lb. when it ruptured. Compute the yield point, the ultimate strength, and the rupture strength.

Ans. 36,800 lb./in.^{2}; 64,300 lb./in.^{2}; 55,000 lb./in.^{2}.

2. Compute the maximum load on a cast-iron bar 1.3 by 3.2 in. in cross section when tested in tension and when tested in compression.

Ans. 83,200 lb.; 332,800 lb.

3. How high could a concrete tower be built before the concrete is crushed at the base by its own weight? *Ans.* 2400 ft.

4. What length of aluminum wire suspended at its upper end would break with its own weight? If suspended in water, what length would break with its own weight? *Ans.* 21,820 ft.; 35,120 ft.

12. Percentage of Elongation and of Reduction of Area.—In tensile tests of steel and wrought iron, as was explained in Art. 11, the test piece is elongated practically uniformly throughout its length until the ultimate strength is reached. After the ultimate strength has been passed, the elongation and the reduction of cross section are localized at the neck.

Both the elongation and the reduction of area of tensile test pieces are measures of the ductility of the metal. Figures 11 to 15 are reproductions of photographs of standard test bars, showing the variation in the amount of elongation and reduction of area in different grades of steel and wrought iron.

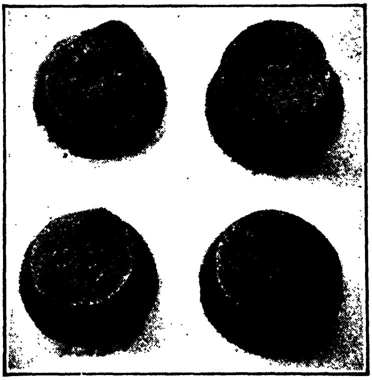


FIG. 11.—Rivet steel.

Since ductility is a very important property of metals for certain kinds of construction, standard specifications, such as those of the American Society for Testing Materials, require a certain percentage of elongation as a minimum. Table VII gives the specifications of the American Society for Testing Materials for wrought iron and several grades of steel.

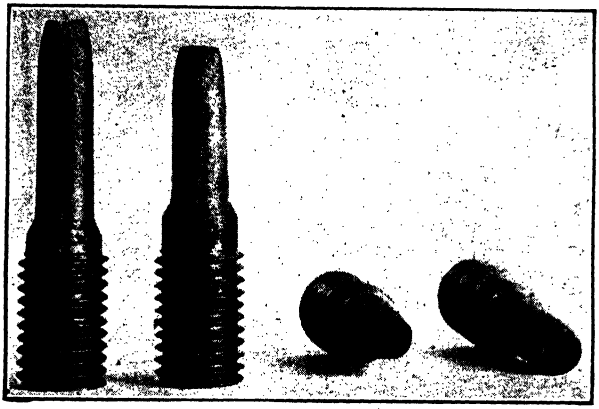


FIG. 12.—Wrought iron.

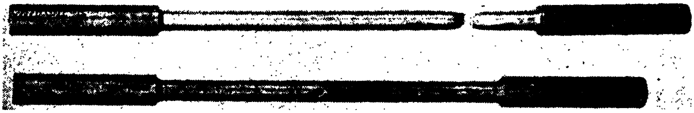


FIG. 13.—Structural steel.

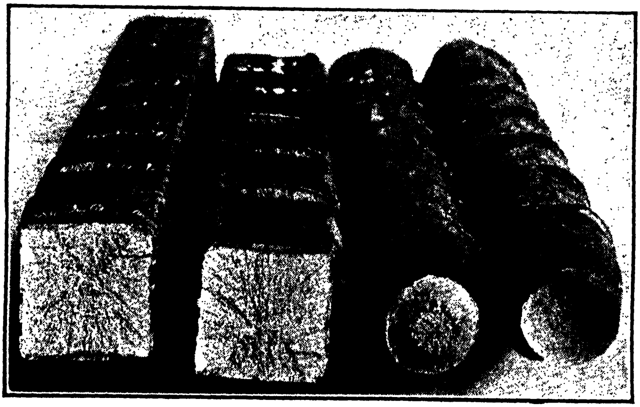


FIG. 14.—Reinforcing steel.

Two standard gage lengths of test pieces have been adopted, an 8-inch length for rods, plates, and shapes from which a test bar 16 to 18 inches long may be cut, and a 2-inch gage length for forgings, castings, bridge pins, rollers, and similar members from which it is not practicable to cut so long a test piece. Rivet rods and small rolled bars are tested in the form in which they are rolled. Test pieces from plates, structural shapes, and bars are the full thickness of the material and are prepared as shown in Fig. 16. If this is impracticable, test pieces may be made as shown in Fig. 17 for such material as heads and flanges of railroad rails, or as shown in Fig. 18 for such material as castings and bridge rollers.

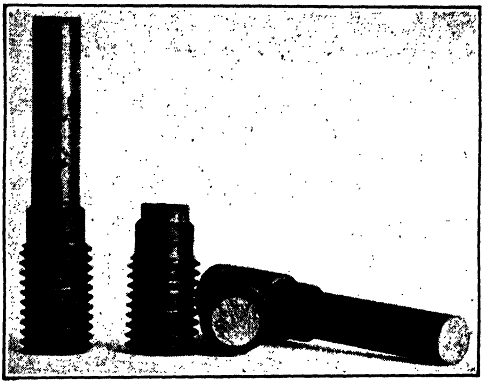


FIG. 15.—Spring steel.

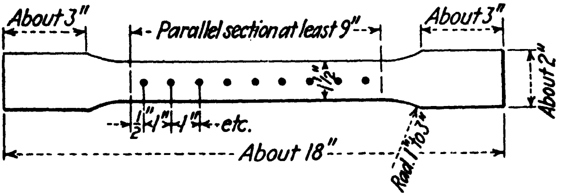


FIG. 16.—A. S. T. M. standard test bar.

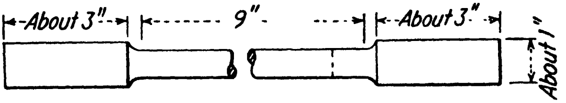


FIG. 17.—Turned test bar.

In making the measurement for the computation of the percentage of elongation, the broken parts of the test piece are fitted together in the same relative position as before fracture and the new length is measured with a scale. The increase in length divided by the original length gives the relative elongation, and this value multiplied by 100 gives the percentage of elongation.

As can readily be seen, a large part of the elongation of a ductile test piece occurs in the neck and along the shoulder just back of the neck. If the fracture occurs close to the end gage mark, a relatively large part of the elongation comes outside of the gage length and so is not measured. Specifications permit that if a test piece fractures outside of the middle third, and if on that account the percentage of elongation is too low, another test piece may be substituted.

The difference between the original area of the cross section and the area of the cross section at the fracture gives the reduction of area. This reduction of area divided by the original area gives the relative reduction of area, and this relative reduction of area multiplied by 100 gives the percentage of reduction of area. It should be noted that the area used is not necessarily the area of the fracture, but is the smallest cross section at the neck. Ductile materials tested in tension usually fail finally by shearing along one or more diagonal planes, giving an area of fracture larger than the smallest cross section. Shearing fracture in tensile tests will be discussed further in Art. 19.

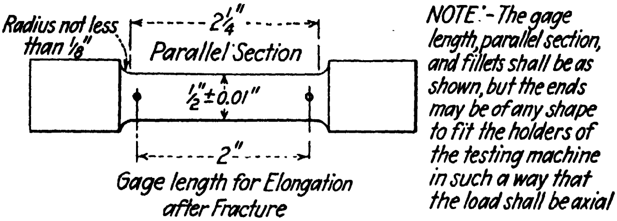

Radius not less than 1/8"
2 1/4"
Parallel Section
1/2"±0.01"
2"
Gage length for Elongation after Fracture
NOTE:- The gage length, parallel section, and fillets shall be as shown, but the ends may be of any shape to fit the holders of the testing machine in such a way that the load shall be axial

FIG. 18.—A. S. T. M. standard test bar for 2-inch gage length.

PROBLEMS

1. A test bar 1.50 in. wide and 0.50 in. thick, with an 8-in. gage length was tested in tension with the following results: final gage length, 10.52 in.; dimensions of smallest cross section at fracture, 0.97 by 0.38 in. Compute the percentage of elongation and the percentage of reduction of area.

Ans. 31.5 per cent; 50.9 per cent.

2. A test bar of structural steel, 1.00 in. wide, 0.19 in. thick, with an 8-in. gage length, was tested in tension with the following results: load at yield point, 6920 lb.; maximum load, 12,600 lb.; dimensions at fracture, 0.70 by 0.13 in. The final dimensions between the 1-in. gage marks are shown in the following table:

INTERVALS BETWEEN GAGE MARKS	FINAL DIMENSION
0–1	1.22
1–2	1.27
2–3	1.18
3–4	1.26
4–5	1.25
5–6	1.55 (neck)
6–7	1.22
7–8	1.18

Compute the yield point, the ultimate strength, the percentage of elongation, and the percentage of reduction of area. Will this bar pass the specifications of the American Society for Testing Materials for structural steel?

Ans. 36,400 lb./in.^{2}; 66,300 lb./in.^{2}; 26.6 per cent; 52.1 per cent.

3. From the data of the table of Problem 2, compute the percentage of elongation for the following intervals: 2–8; 3–7; 4–6.

Ans. 27.3 per cent; 32 per cent; 40 per cent.

4. A tension test piece of structural steel with the shape and dimensions shown in Fig. 18 held a load of 7000 lb. at the yield point and a load of 15,000 lb. at its maximum. The final gage length was 2.48 in. In what items does this test bar fail to pass the specifications of the American Society for Testing Materials?

13. Working Unit Stresses and Factor of Safety.—As previously stated, it has been found by experiment that a stress greater than the elastic limit produces a permanent deformation of the material, and a small number of repetitions of such stress will usually cause failure. It has also been found by experiment that, even though the stress is kept below the elastic limit, a large number of repetitions of the loading may cause failure, especially if the stress is reversed. Such fractures apparently begin with small cracks which spread progressively on account of the greater stress at the base of the crack, and after failure the fracture even of ductile material has the appearance of a brittle material. Such failures are called *fatigue* failures, but the term *progressive fracture* would be more descriptive.

The maximum unit stress to which a material may be subjected repetitively an indefinite number of times without failure is called the *endurance limit,* a value somewhat below the elastic limit.

Safe working unit stresses for machine members or structural members subjected to repetitive stresses must be less than the endurance limit if failure is to be avoided, while for all other members they must be less than the elastic limit. Tables of allowable unit stresses give values which some authority has determined to be safe for the conditions of loading stated. Tables III and IV, at the end of this volume, give extracts from the New York Building Laws and the American Bridge Company specifications. Table V gives extracts from the table of safe working stresses for timber recommended by the Forest Products Laboratory, Madison, Wisconsin, and approved by the Building Code Committee of the U. S. Department of Commerce. For other grades of timber and for timber exposed to the weather, certain reductions in the allowable stresses must be made. For complete data on these reductions, see the American Society for Testing Materials Standards, 1933, Part II.

Table XV gives the safe working stresses for steel recommended by the Building Code Committee. This committee recognizes two grades of steel: (1) steel considered acceptable, but of which the origin and physical characteristics are not definitely determined, and (2) steel which conforms to the standard specifications of the American Society for Testing Materials for structural steel for buildings. Table XVI gives the specifications of the American Institute of Steel Construction for steel which conforms to the American Society for Testing Materials standards.

The term, *factor of safety* of a material, as commonly used, is the ratio of its ultimate strength to its working stress. As noted above, the value of the safe working stress of a material should be based upon the endurance limit or the elastic limit, rather than upon the ultimate strength, and this method is coming into use more and more. However, even though the endurance limit or the elastic limit is used to determine the safe working stress, the value of its "factor of safety" in terms of the ultimate strength is still in common use.

In addition to the requirement that safe working stresses must be within the endurance limit or the elastic limit, the possible deterioration of a member due to wear, corrosion, rotting, or other cause during the normal life of the structure must always be taken into consideration.

PROBLEMS

1. If the allowable stress for steel eyebars is 36,000 lb./in.^{2}, how many bars 12 in. wide and 2 in. thick will be necessary to carry a load of 1500 tons? *Ans.* Four.

2. By the New York Building Laws, what must be the size of a square limestone pier to support a column load of 1800 tons? *Ans.* 6 ft.

3. By means of the specifications of the American Bridge Company, get the commercial size of a round tie-rod to carry a load of 28,000 lb.

Ans. 1½ in.

4. Western-hemlock floor joists 3 in. wide, 20 in. deep, and 20 ft. long, spaced 18 in. center to center, support 2-in. flooring of the same material and a superimposed floor load of 250 lb./ft.^{2} of floor area. If the joists are supported on a concrete wall at each end, what must be the thickness of the walls to furnish the necessary supporting area? Western hemlock weighs 28 lb./cu. ft. *Ans.* 4.37 in.

5. With a factor of safety of 4, what load may be placed upon a wrought-iron bar 3 in. in diameter which just passes the specifications of the American Society for Testing Materials? *Ans.* 84,820 lb.

6. A hollow rectangular concrete pier is to be 3 by 4 ft. outside dimensions. If it is to carry a load of 500,000 lb. with a factor of safety of 6, what must be the inside dimensions if all walls are the same thickness?

Ans. 17¾ by 29¾ in.

14. Actual Stresses.—In all of the preceding discussion, unit stresses have been computed by dividing the total load by the original area of the cross section. This is the usual method, and stresses are always so meant if no qualifying statement is made.

For purposes of comparison, however, it is sometimes desirable to compute the actual unit stresses of ductile materials in tension by dividing the total load by the actual area of the cross section at that time. For stresses below the elastic limit there is no appreciable difference between the actual unit stress and the stress as usually computed, since the elastic reduction in area is very small. After the elastic limit has been passed, however, the larger permanent deformations occur which may readily be measured. For values of the unit stress between the elastic limit and the ultimate strength, the area of the cross section at any given load may be computed by dividing the volume, which is practically a constant, by the length at that load. That is, if A and l are the original cross-section area and length respectively, A' and l' the area and length respectively at any given load, and V the volume,

$$V = Al = A'l'. \quad \text{(Approx.)}$$

$$A' = \frac{Al}{l'}. \quad \text{(Approx.)}$$

As soon as the ultimate strength has been passed and the test piece begins to neck, the relation stated above is no longer true, and the only method of determining the area at the smallest part of the neck is by direct measurement. On account of the rapidity of change both in the load and in the area of cross section at the neck, these measurements are extremely difficult to make. The final measurement of the cross section at the fracture may be made accurately, but the exact load which the test piece was holding just previous to fracture is difficult to determine. It is seen, then, that accurate values of the actual unit stresses may be obtained below the ultimate strength, but not between the ultimate strength and the rupture strength.

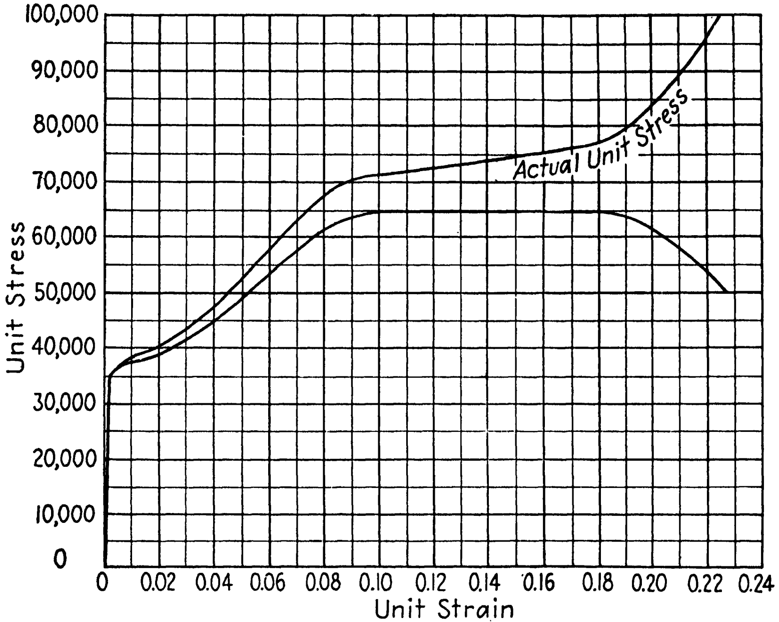


FIG. 19.—Stress-strain diagram of steel. Actual stress.

It should be noted that although there is sometimes a reduction in the load just after the yield point, and in the unit stress as ordinarily computed, there is no reduction in the actual stress. Figure 19 shows the actual stress-strain curve compared with the curve as usually drawn.

15. Effect of Stresses above the Elastic Limit.—As has already been mentioned in Art. 5, a bar which is stressed to a value exceeding the elastic limit receives a deformation, some of which remains as a permanent set after the stress has been removed.

In the case of nonductile materials, such overstressing causes permanent injury, and a few repetitions of such loading are usually sufficient to cause rupture.

If such a material as wrought iron, mild steel, or medium steel is stressed to a value exceeding the yield point, its physical properties are changed markedly. Its yield point is raised and its deformation above the yield point is decreased. It becomes harder, stiffer, and much less ductile. Cold-drawn steel or wrought-iron wire, deformed reinforcing bars, and cold-rolled steel shafting are examples of such overstressed material.

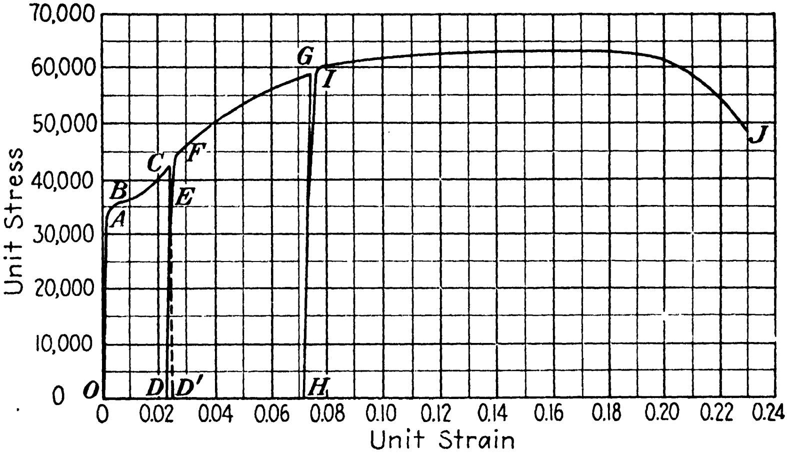


FIG. 20.—Stress-strain diagram of steel. Interrupted loading.

Figure 20 represents the stress-strain diagram for a bar of medium steel stressed in tension. Point A on the curve represents the elastic limit and point B represents the yield point. When the stress had reached the value indicated by point C, the loading was interrupted and the stress was reduced to zero, as indicated at point D. As the stress was reduced, the strain decreased by an amount $D'D$ while the part OD remained as a permanent set. The cross section was permanently reduced. Line CD is practically parallel to line OA.

The load was immediately applied again and up to point E the curve practically coincided with the former curve CD, showing that the modulus of elasticity of the material remained unchanged. After passing the elastic limit at E, the stress reached a new yield point value at F, slightly higher than the former stress at

C and considerably higher than the former value of the yield point at *B*.

A second interruption of the loading, when the stress had reached a value indicated by point *G*, gave a similar interruption of the curve. If the bar had been loaded continuously, the stress-strain diagram would have been very similar to the upper part of the diagram as shown.

In computing the values of the stresses used in the diagram, Fig. 20, the original area was used. If after either interruption the new cross-section area had been used and the stresses computed for that area, the values of both the elastic limit and the yield point would have been higher.

CHAPTER II

STRESSES AND STRAINS IN SHEAR

16. Shearing Stresses.—Shearing forces upon a bar are forces which cause or tend to cause one part of a bar to slip with respect to the part next to it. In order that there shall be shearing stress along the cross section of a bar such as that shown in Fig. 21, the external loads must necessarily act in opposite directions and in different planes of action, normal to the bar.

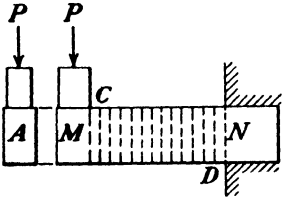


FIG. 21.

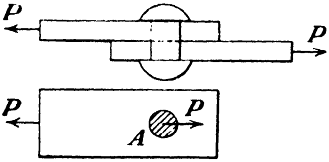


FIG. 22.

Only the part of the bar between these planes is subject to the shearing action. At each cross section of the bar between *M* and *N* there is a tendency for the part of the bar on the left of that section to slip downward with respect to the part on the right side of that section. If *A* is the area of the cross section of the bar, the average unit shearing stress is $S_s = P/A$. The bar *MN* is really a beam in which longitudinal tension and compression are developed, and these stresses affect the unit intensity of the shearing stress over the cross section as will be explained in Art. 54.

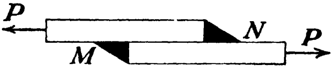


FIG. 23.

If two plates are held together by a rivet, Fig. 22, and equal and oppositely directed forces *P*, *P* are applied, the cross section of the rivet between the plates is subjected to a shearing stress, the average unit intensity of which is $S_s = P/A$.

If two plates are welded together by means of fillet welds, as shown in Fig. 23, the area at the base of each fillet along the plane MN is subjected to shearing stress. If A is the area of contact of the two fillets with the plates along the plane MN, the average unit intensity of the shearing stress is $S_s = P/A$.

If a force P is applied to a punch to punch a hole through a plate as shown in Fig. 24, the metal around the circumference of the slug MN must fail in shear as the slug is separated from the rest of the plate. If d is the diameter of the punch and t the thickness of the plate, the area sheared is $A = \pi dt$, and the average unit shearing stress is $S_s = P/A$.

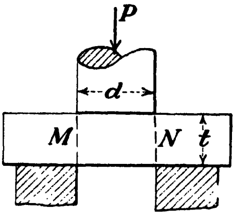


FIG. 24.

Even in the case of the rivet between two plates, of the fillet weld between two plates, and of the slug punched out of a plate, there is a slight bending or beam action, and on this account the shearing stress is not uniform over the cross section. It is customary, however, to neglect this small variation in these three cases and make computations as though the stress were uniform.

A bar in torsion is one such as that shown in Fig. 25 to which couples are applied at each end to twist it around its own axis. In this case pure shearing stresses are developed on each cross section between the twisting couples, since the couples cause neither bending nor direct tension or compression. Each of the thin plates or disks of which the bar may be assumed to consist has equal and opposite twisting couples on its two faces. These twisting couples consist of shearing forces which at every point are acting normal to the radius in the plane of the cross section. The distribution of these shearing forces will be discussed in Chap. IV.

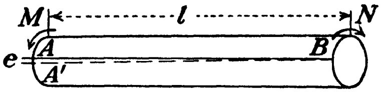
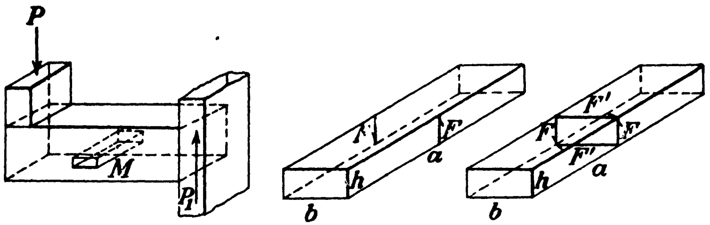


FIG. 25.

It is evident from static conditions that a single set of shearing forces cannot exist alone upon a rigid body in equilibrium, since together they form a couple. Figure 26(a) represents a bar subjected to two external shearing loads P, P_1. The elementary parallelepiped at M, shown to a larger scale in Fig. 26(b), has acting upon it the two equal and opposite shearing forces F, F, each of which is equal to $S_s ah$. The moment of this couple is $S_s ahb$. The necessary balancing couple cannot be furnished by either tensile or compressive forces, but only by an induced shearing couple consisting of forces F', F', in the horizontal surfaces as shown in Fig. 26(c). If the unit shearing stress in each horizontal surface is S_s', the total force F' is $S_s'ab$, and the moment of the induced couple becomes $S_s'abh$. Since the two couples are equal and opposite,

$$S_sahb = S_s'abh.$$
$$S_s = S_s'.$$



(a) (b) (c)

FIG. 26.

A direct shearing stress cannot exist alone, but induces a shearing stress of equal unit intensity upon a plane normal to the plane of direct shear.

PROBLEMS

1. If the rivet shown in Fig. 22 is $1\frac{3}{16}$ in. in diameter and the allowable average unit shearing stress is 8800 lb./in.^{2}, what is the allowable force P?
Ans. 9750 lb.

2. If the ultimate strength of a $1\frac{1}{2}$-in. bolt is 60,000 lb./in.^{2} and the ultimate shearing strength of the head of the bolt is 40,000 lb./in.^{2}, what must be the thickness of the head of the bolt in order to resist being stripped from the bolt?
Ans. $\frac{9}{16}$ in.

3. Compute the force required to punch a $\frac{9}{16}$-in. hole through a $\frac{1}{2}$-in. plate, if the ultimate shearing strength of the plate is 45,000 lb./in.^{2}. Compute the average unit compressive stress in the punch.
Ans. 39,800 lb.; 160,000 lb./in.^{2}.

4. If S_c is the allowable unit compressive stress in a punch, S_s the ultimate unit shearing stress in a plate, t its thickness, and d the diameter of the punch, show that the maximum thickness of the plate through which a hole of given diameter may be punched is given by the expression, $t = dS_c/4S_s$.

5. Show that the expression $d = 4S_st/S_c$ gives the minimum diameter of the hole that can be punched through a given plate.

17. Shearing Strains and Shearing Modulus of Elasticity.—

A bar of elastic material which is subjected to a pair of shearing forces such as P, P', Fig. 27, is distorted in shape, since each thin plate of which the bar may be assumed to be composed is displaced slightly downward with respect to the one on the right of it. The total shearing deformation in the length l is e, and the unit shearing strain, or the shearing strain at a distance of unity from the fixed plane, is given by the expression,

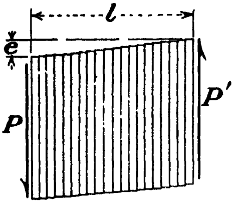


FIG. 27.

$$\delta_s = \frac{e}{l}.$$

The bar shown in Fig. 27 is really a beam in which it is not possible to develop the shearing stresses and strains indicated without developing also tension in the upper fibers, compression in the lower fibers, and a deflection of the left end with respect to the right end due to the bending. It is impossible, therefore, to determine the values of δ_s experimentally from such a bar, since the deflection is composed of the two parts, that due to shear and that due to bending.

In the bar which is twisted around its own axis, Fig. 25, the deformation is all due to shear. Since the bar is elastic, each of the thin plates or disks of which the bar may be assumed to consist, twists on its axis with respect to those on each side of it. The element AB, after the bar is twisted, takes the new position $A'B$. At the outer surface the deformation is from A to A', the distance e. The unit shearing strain is given by the expression,

$$\delta_s = \frac{e}{l}.$$

The modulus of elasticity in shear, denoted by E_s, is the ratio of the unit shearing stress in a bar to the resulting shearing strain.

$$E_s = \frac{S_s}{\delta_s}.$$

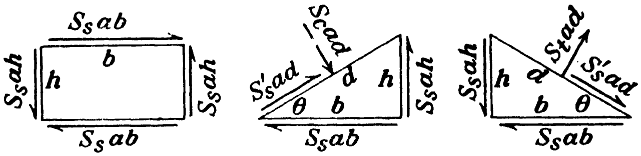
For homogeneous materials for which Poisson's Ratio $m = 0.25$ to 0.30, the value of E_s is about two-fifths of the value of E. Table VIII at the end of this volume gives average values of E_s for several engineering materials.

PROBLEMS

1. A solid cylindrical shaft is 4 in. in diameter and 80 ft. long. If E_s = 11,500,000 lb./in.^{2}, what unit shearing stress is caused when one end of the shaft is twisted 15° with respect to the other end? *Ans.* 6280 lb./in.^{2}.

2. The wire of a torsion balance is 0.2 in. in diameter and 4 ft. long. If E_s = 12,000,000 lb./in.^{2} and allowable S_s = 20,000 lb./in.^{2}, through what total angle may the balance be rotated? *Ans.* 91° 41′.

18. Tension and Compression Caused by Shear.—In Fig. 28(*a*) is shown the end view of the elementary parallelepiped from the bar in shear, Fig. 26, with both the direct and the induced shearing forces acting. The right lower half of this parallelepiped is shown as a free body in Fig. 28(*b*). If d is the length of the diagonal, the forces upon the diagonal surface are the shearing force $S_s'ad$ and the compressive force S_cad. Since the free body is in equilibrium, the summation of the forces in a direction normal to the diagonal surface must equal zero.



(a) (b) (c)

FIG. 28.

$$S_cad = S_sah \cos\theta + S_sab \sin\theta.$$

$$S_c = S_s\frac{h}{d}\cos\theta + S_s\frac{b}{d}\sin\theta.$$

$$S_c = 2S_s \sin\theta \cos\theta.$$

$$S_c = S_s \sin 2\theta.$$

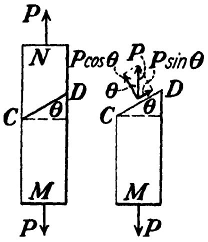
This expression becomes a maximum when $\sin 2\theta = 1$. For this value, $2\theta = 90°$, $\theta = 45°$, and $S_c = S_s$.

Upon the other diagonal plane the stress is tension, as shown in Fig. 28(*c*). This stress also reaches a maximum value of $S_t = S_s$ when $\theta = 45°$.

A direct shearing force and its induced shearing force cause a resultant compressive stress of the same unit intensity on a diagonal plane at an angle of 45° *with the plane of either shearing force, and a resultant tensile stress of the same unit intensity on the other diagonal* 45° *plane.*

The effect of the resultant compressive stress caused by shear will be discussed in Art. 126. It causes diagonal column action in the webs of plate girders and I-beams. The effect of the resultant tensile stress caused by shear will be discussed in Art. 86. It causes diagonal tension in reinforced concrete beams.

19. Shearing Stresses Caused by Direct Tension or Compression.—In Fig. 29(*a*) is shown a bar *MN* under the action of tensile forces *P*, *P*. Let the bar be separated into two parts by the diagonal plane *CD*, at the angle *θ* with the cross section of the bar, and let the lower part of the bar be taken as a free body, as shown in Fig. 29(*b*). At the section *CD* the force *P*, equal and opposite to the force *P* at the bottom, is resolved into its two components, normal and parallel to the section *CD*. If *A* is the area of the cross section of the bar, the area of the diagonal section *CD* is $A/\cos\theta$. The unit shearing stress upon the plane *CD* is given by the expression,



(a) (b)

FIG. 29.

$$S_s = \frac{P}{A}\sin\theta\cos\theta = \frac{P}{2A}\sin 2\theta.$$

This expression becomes a maximum when $\sin 2\theta = 1$. For this value, $2\theta = 90°$ and $\theta = 45°$.

$$\text{Max. } S_s = \frac{P}{2A}.$$

It is evident that if force *P* were a force in compression, the same value of the induced shearing stress would be obtained.

Direct tensile or compressive forces induce a diagonal shearing stress whose maximum unit intensity is one-half that of the direct stress and whose plane is inclined at an angle of 45° *with the cross section.*

The relation just derived is true for all values of the stress up to the ultimate strength. Ductile materials in tension just after passing the elastic limit often show diagonal lines on the surface, indicating planes of slip in shear. Ductile materials in tension will also usually fracture partly or wholly along a diagonal plane, as shown in Fig. 30. In circular bars this causes the well-known "cup" fracture. Timber, stone, brick, neat cement cubes, concrete blocks, and similar materials tested to destruction in compression usually show well-defined diagonal shear fracture. The structure of the material will sometimes change slightly the direction of the plane of failure. Figure 31 shows typical failures in shear caused by compression.

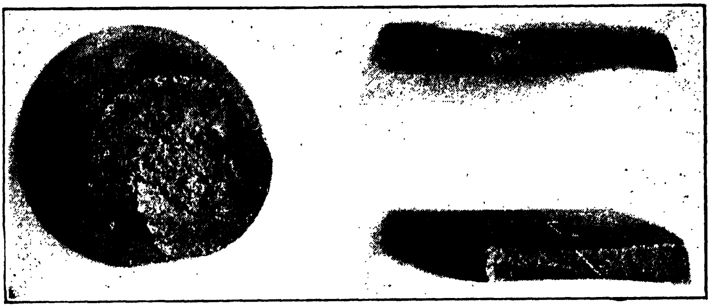


FIG. 30.—Diagonal shearing fracture caused by tension.

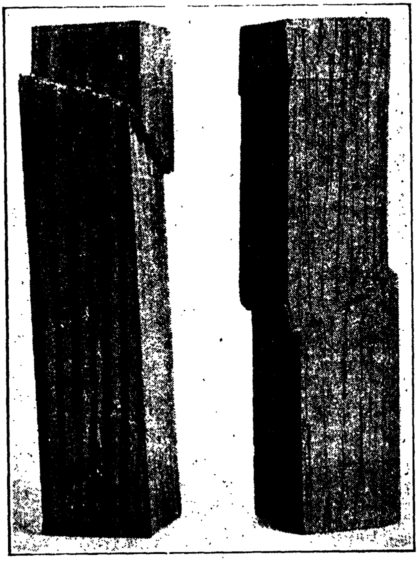


FIG. 31.—Diagonal shearing fracture caused by compression.

PROBLEMS

1. A steel bar $\frac{1}{2}$ in. square is loaded with 10,000 lb. in tension. Compute the unit shearing stresses on diagonal planes at angles of 30, 40, 45, and 50° with the cross section of the bar.

Ans. 17,320 lb./in.^{2}; 19,696 lb./in.^{2}; 20,000 lb./in.^{2}; 19,696 lb./in.^{2}.

2. A steel bar 1.06 by 0.17 in. failed in diagonal shear on a plane at an angle of 42° with the cross section when subjected to a tensile load of 10,050 lb. Compute the amount of the unit shearing stress on this plane. Compute the amount of the theoretical maximum unit shearing stress.

Ans. 27,720 lb./in.^{2}; 27,885 lb./in.^{2}.

3. After rupture, the dimensions of the diagonal area of fracture of the bar described in Problem 2 above, were 0.71 by 0.18 in. Compute the actual unit shearing stress on the area of fracture. *Ans.* 52,620 lb./in.^{2}.

4. Using the values given in Table V, compute the maximum angle that the grain of a short block of cypress loaded in compression may have with its length. If the grain is at an angle of 15° with its length, by what percentage must the allowable load be reduced? *Ans.* 5° 15′; 63.5 per cent.

CHAPTER III

RIVETED AND WELDED JOINTS

20. Stresses in Simple Riveted Joints.—If two plates are held together by a rivet as shown in Fig. 32 and a load is applied in tension, the force is transmitted from one plate to the other through the rivet. At the rivet the plates lap over each other, so the joint is called a *lap joint.* Let p be the width of the plate, t its thickness, and d the diameter of the rivet. The unit tensile stress in the gross area, as at section AB, is $S_t = P/A = P/pt$. At the section CD through the rivet hole, the cross-section area is only $(p - d)t$, and the average unit tensile stress in the net area is $S_t = P/(p - d)t$.

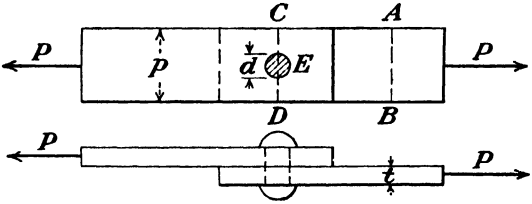


FIG. 32.

It has been found by experiment that the intensity of stress at the edges of the rivet hole is considerably higher than the average value, but in ordinary commercial work this variation in stress is commonly neglected. In this chapter the average unit tensile stress will be used.

On the side of the rivet at E in the upper plate and on the opposite side of the rivet in the lower plate, force is transmitted from the plate to the rivet in compression or bearing. On account of the curved surface of the rivet and the rivet hole, the stress is variable in direction and in unit intensity, but if the rivet fills the hole, the unit intensity of the bearing stress cannot exceed the amount that it would be if the total pressure were acting upon the projected area td. The unit stress in bearing is therefore computed as $S_c = P/td$.

At the cross section of the rivet in the plane between the two plates, the pull from one plate is transmitted across into the other by means of shear in the rivet. This shearing stress is assumed to be uniformly distributed over the cross section, so the unit shearing stress is $S_s = 4P/\pi d^2$.

There is also a shearing stress in the plate at the sections *F* and *G*, Fig. 33. The value of the average unit shearing stress at these sections is obtained by dividing the load *P* by twice the area between the edge of the plate and the edge of the rivet hole. Added strength of the plate in shear may be obtained easily by placing the rivet hole farther from the edge of the plate. Except in unusual cases, therefore, no examination of the strength of the plate in shear is necessary if the joint is designed according to some standard specifications for riveting.

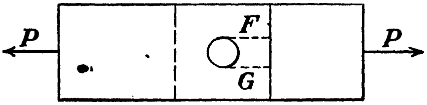
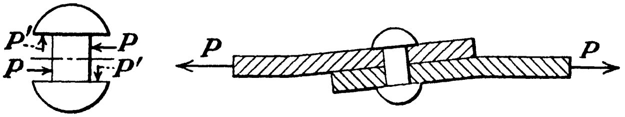


FIG. 33.

Since forces *P*, *P*, Fig. 32, are not in the same plane, there are bending and tensile forces exerted upon the rivet in addition to the bearing and shearing forces. If the resultant of the pressure from each plate upon the rivet is assumed to act midway between the surfaces of the plate, the moment upon the rivet is *Pt*. This moment must be balanced by an equal and opposite moment consisting of bearing forces *P′*, *P′*, exerted by the plate upon the head of the rivet, as shown in Fig. 34(*a*). These forces cause tension in the rivet.



(a) (b)

FIG. 34.

A bending force is likewise exerted upon each plate at the rivet, tending to bend the plates as shown in Fig. 34(*b*). If the bending stress in the plate exceeds the elastic limit of the material, the plate will be permanently deformed, causing some redistribution of the stress upon the rivet.

A rivet which is driven tight while hot will tend to contract upon cooling and this contraction, if prevented by the plates, will cause some tensile stress in the rivet. There is also a frictional force developed between the plates due to this tension in the rivet. It is customary in the computation of stresses in riveted connections to neglect both the initial tension in the rivet and the frictional resistance between the plates, since accuracy of computation is extremely difficult and both of the forces are greatly reduced as soon as initial slip occurs.

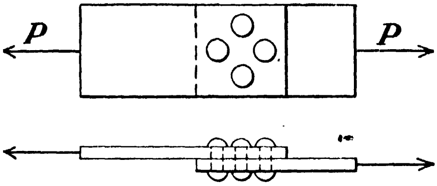


FIG. 35.

In Fig. 35, a lap joint is shown in which the two plates are connected by four rivets. Each one of the rivets is assumed to transmit one-fourth of the stress from one plate to the other. The net cross-section area of each plate through the first rivet hole must carry the total load *P*. The net cross-section area of each plate through the two rivet holes in the middle row carries only three-fourths of the load *P*, since one-fourth of the load has been transmitted through the first rivet into the other plate. The four rivet areas in the plane between the two plates must carry the total load in shear and each rivet area is assumed to carry one-fourth of the load. Similarly, each of the four rivets must carry one-fourth of the load *P* in bearing on the side of the rivet.

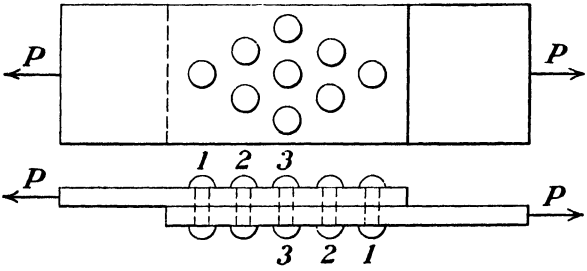


FIG. 36.

In Fig. 36, the two plates are connected by nine rivets, and each rivet is assumed to transmit $\frac{1}{9}$ of the load from one plate to the other. The net cross-section area of each plate in row 1 must carry the total load P. The net cross-section area of each plate in row 2 carries only $\frac{8}{9}$ of P since $\frac{1}{9}$ of P has been transmitted by the first rivet into the other plate. The net cross-section area of each plate in row 3 carries only $\frac{6}{9}$ of P, since $\frac{3}{9}$ of P has been transmitted by the three rivets into the other plate. The nine rivet areas in the plane between the two plates must carry the total load in shear and each rivet area is assumed to carry $\frac{1}{9}$ of the load. Similarly each rivet is assumed to carry $\frac{1}{9}$ of the load in bearing on the side of the rivet.

In general, sizes of undriven rivets vary by eighths of an inch. Rivet holes are usually made $\frac{1}{16}$ inch larger in diameter than the cold rivet. When used, the rivets are heated red hot and in the process of riveting are enlarged to fill the holes completely. If the corresponding holes in the plates are concentric and a perfect job of riveting is done, the diameter of the hole and the finished rivet should be used in computations of shearing and bearing stresses. If the corresponding holes are not concentric, the original or nominal diameter of the rivet should be used.

If rivet holes are punched and the plate is not subsequently annealed, the material at the edge of the hole is injured to a certain extent. On this account specifications commonly require that for punched holes the net section of the plate in tension shall be obtained by deducting from the width of the plate the nominal diameter of the rivet plus $\frac{1}{8}$ inch. If rivet holes are drilled, or are subpunched and reamed to size, the actual net section is used.

For lap joints with plates thicker than $\frac{1}{2}$ inch, the bending action mentioned above becomes excessive and butt joints are customarily used. Figure 37 represents two main plates *a*, *a* butted together and attached to each other by means of two cover plates or butt straps *b*, *b*, to form a butt joint. It is necessary to examine the stresses on only one side of the middle.

Each rivet is said to be acting in double shear, since it would be necessary to shear two cross-section areas of each rivet in order to separate the main plate from the cover plates or butt straps. The bearing of the main plate on the rivets, since it is between the two butt straps, is called *enclosed bearing*, in contradistinction to plain bearing as in the butt straps and in the plates of lap joints. It is customary to permit higher allowable values for enclosed bearing than for plain bearing. For this reason and also because there are usually more rivets in the last row of rivets in a butt joint than in the first row, the butt straps must be thicker than one-half of the thickness of the main plate.

The stresses in tension in the main plate are computed the same as they are in the lap joint. The stress in tension in the cover plates is a maximum at the inner row where all of the force must be carried by the area remaining after the three rivet holes are cut out.

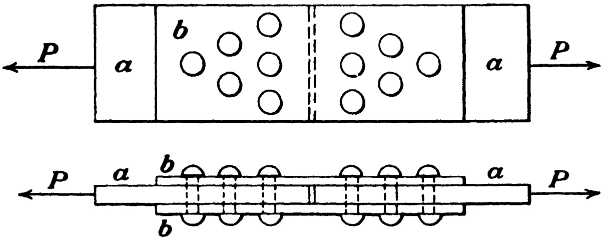


FIG. 37.

EXAMPLE 1

In Fig. 35, let each plate be 5 in. wide and $\frac{1}{4}$ in. thick, the rivets $\frac{1}{2}$ in. in diameter in $\frac{9}{16}$-in. punched holes, and the load 8000 lb. Compute the unit tensile stress in the gross area, the unit tensile stress in the net area in the first row of rivets and in the second row of rivets, the unit shearing stress in the rivets, and the unit bearing stress on the side of the rivets. Use the nominal size of the rivets.

Solution.—Tension, gross area. Gross area = 5 × 0.25 = 1.25 sq. in.

$$S_t = \frac{8000}{1.25} = 6400 \text{ lb./in.}^2.$$

Since the holes are punched, the diameter of hole deducted is $\frac{5}{8}$ in.

Tension, first row. Net area = 4.375 × 0.25 = 1.094 sq. in.

$$S_{t_1} = \frac{8000}{1.094} = 7310 \text{ lb./in.}^2.$$

Tension, second row. Net area = 3.75 × 0.25 = 0.9375 sq. in.

$$S_{t_2} = \frac{6000}{0.9375} = 6400 \text{ lb./in.}^2.$$

Shear. The area of the rivets in shear is 4 × 0.1963 = 0.7854 sq. in.

$$S_s = \frac{8000}{0.7854} = 10{,}190 \text{ lb./in.}^2.$$

Bearing. The area of the rivets in bearing is $4 \times 0.25 \times 0.5 = 0.5$ sq. in.

$$S_c = \frac{8000}{0.5} = 16{,}000 \text{ lb./in.}^2.$$

EXAMPLE 2

In Fig. 37, let the main plates be 10 by $\frac{3}{4}$ in., the butt straps 10 by $\frac{1}{2}$ in., the rivets 1 in. in diameter in $1\frac{1}{16}$-in. drilled holes, and the load 80,000 lb. Compute the unit tensile stress in the main plates at the first two rows of rivets, and in the butt straps at the third row, the unit shearing stress in the rivets, the unit bearing stress on the sides of the rivets both in the main plate and in the butt straps. Use the nominal size of the rivets.

Solution.—Since the holes are drilled, the diameter of the hole deducted is 1.0625 in.

Tension, first row. Net area $= 8.9375 \times 0.75 = 6.7$ sq. in.

$$S_{t_1} = \frac{80{,}000}{6.7} = 11{,}930 \text{ lb./in.}^2.$$

Tension, second row. Net area $= 7.875 \times 0.75 = 5.91$ sq. in.

$$S_{t_2} = \frac{5}{6} \times \frac{80{,}000}{5.91} = 11{,}280 \text{ lb./in.}^2.$$

Tension, butt straps, third row. Net area $= 6.8125 \times 0.5 \times 2 = 6.8125$ sq. in.

$$S_{t_3} = \frac{80{,}000}{6.8125} = 11{,}720 \text{ lb./in.}^2.$$

Shear. There are six rivets, all in double shear, so the area in shear is $2 \times 6 \times 0.7854 = 9.425$ sq. in.

$$S_s = \frac{80{,}000}{9.425} = 8490 \text{ lb./in.}^2.$$

Bearing, main plate. The bearing area in the main plate is $6 \times 0.75 \times 1 = 4.5$ sq. in.

$$S_c = \frac{80{,}000}{4.5} = 17{,}800 \text{ lb./in.}^2.$$

Bearing, butt straps. The bearing area in the butt straps is $6 \times 2 \times 0.5 \times 1 = 6$ sq. in.

$$S_c' = \frac{80{,}000}{6} = 13{,}330 \text{ lb./in.}^2.$$

PROBLEMS

1. Two plates each 3 by $\frac{1}{2}$ in. are fastened together in a lap joint by a $\frac{7}{8}$-in. rivet in $\frac{15}{16}$-in. punched holes. Compute the unit tensile stress in the plates in the gross section and in the net section, the unit shearing stress in the rivet, and the unit bearing stress on the side of the rivet due to a tensile load of 8000 lb.

Ans. 5330 lb./in.^{2}; 8000 lb./in.^{2}; 13,300 lb./in.^{2}; 18,300 lb./in.^{2}.

2. In Fig. 35, let each plate be 5 by ½ in., and the rivets ¾ in. in diameter in 13/16-in. punched holes. When the plates are loaded in tension so that the shearing stress in the rivets is 9000 lb./in.^{2}, what is the unit tensile stress in each row of rivets and the unit bearing stress on the side of the rivets?

Ans. S_{t_1} = 7710 lb./in.^{2}; S_{t_2} = 7350 lb./in.^{2}; S_c = 10,600 lb./in.^{2}.

3. In Fig. 36, let each plate be 6 by ⅜ in., and the rivets ½ in. in diameter in 9/16-in. drilled holes. Compute all of the governing stresses due to a load of 20,000 lb.

Ans. S_{t_1} = 9830 lb./in.^{2}; S_{t_2} = 9730 lb./in.^{2}; S_{t_3} = 8250 lb./in.^{2}; S_s = 11,330 lb./in.^{2}; S_c = 11,850 lb./in.^{2}.

21. Deformations in Riveted Joints.—When a riveted joint is stressed so that it fails, such failure may occur in any one of several different ways. The plate may stretch and neck at the rivet hole and finally fail in tension, as shown in Fig. 38(*a*). The plate may compress behind the rivet, as shown in Fig. 38(*b*). The plate may shear between the rivet hole and the edge of the plate, as shown in Fig. 38(*c*). On account of the bulging of the plate behind the rivet, the plate may fail in tension at right angles to the direction of the load, as shown in Fig. 38(*d*). The rivet may fail in shear, as shown in Fig. 38(*e*). In the case of a lap joint, the plates may be bent as shown in Fig. 34(*b*). Manifestly, before failure occurs, it is possible for the joint to be deformed in several of these ways at the same time.

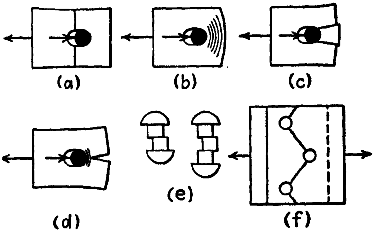


Fig. 38.

If there are two or more rows of rivets and the rivets are staggered, the line of fracture in tension may zigzag back and forth from one line of rivets to the other, as shown in Fig. 38(*f*). In this case the failure is partly in tension and partly in shear.

22. Stresses in Structural Riveted Joints.—In modern structural work the plates and steel shapes which compose beams, trusses, towers, and similar structures are usually held together by rivets. Figure 39 represents a portion of a column composed of two channels and two plates. The only requirement for the rivets is that they shall be spaced closely enough to make the parts of the column act together as a unit. There is no method of computing the stresses caused in the rivets by a load on the column. Figure 40 represents a joint on the upper chord of a roof truss. The stress in the member *c* must be transmitted to the plate *b* by means of the rivets.

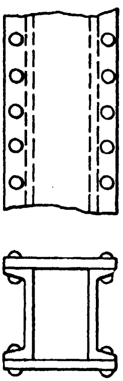


FIG. 39.

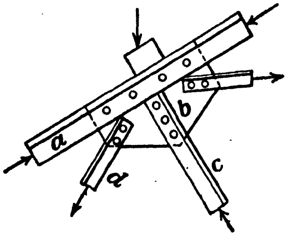


FIG. 40.

The rivets are stressed both in compression and in shear. If the member *c* is composed of one angle, the rivets are acting in single shear; that is, only one cross-section area is subjected to a shearing stress. If the member *c* is composed of two angles, the rivets are acting in double shear. There is less stress in the member *d*, so fewer rivets are needed to connect it to the plate.

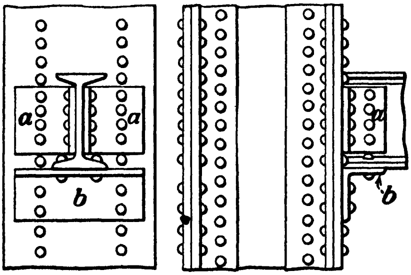


FIG. 41.

Figure 41 represents a portion of a column composed of a web plate, four angles and two flange plates, to which is attached an I-beam girder. The connection consists of two angle bars, *a*, *a*, riveted to the web of the I-beam, and an angle bracket *b* upon which the girder rests. The rivets transmit the end reaction of the girder to the column and are stressed both in shear and in compression. Similar connections, but with the bracket angle omitted, are used to fasten the floor stringers to the web of the girder. In the floor systems of bridges, similar but more elaborate connections are used to fasten the stringers to the floor beams, and the floor beams in turn to the trusses.

In addition to the direct vertical end reaction just mentioned, the loading upon the girder may cause a bending moment in the connection, the amount of the moment depending upon the rigidity of the connection. Upon the upper rivets through the web of the I-beam there will be caused an additional shearing stress horizontally, the shear from the web of the I-beam acting on the rivets away from the column. Upon the two rivets through the flange of the I-beam, and usually upon the lower rivets through the web, depending upon the spacing, there will be shearing forces in the opposite direction. Upon the upper rivets connecting the angles to the column, this bending moment causes a direct tension. At the lower part of the connection, the corresponding compression acts directly between the angles and the column.

Except in special cases, the bending moment mentioned in the preceding paragraph is neglected. In steel handbooks, tables of safe loads on I-beams fastened with such connections are computed for vertical end reactions only. In case it is assumed that the rigidity of the connection is sufficient to cause moment at the end of the beam, the specifications of the American Institute of Steel Construction require that the allowable tensile stress in the rivets shall not exceed a value of 13,500 pounds per square inch on the nominal size of the rivet.

In structural work it is usually difficult to have the punching done with sufficient accuracy to insure that corresponding rivet holes are concentric. The following is the specification of the American Institute of Steel Construction.

> The accuracy of punching shall be such that for any group of holes when assembled, 75 per cent shall admit a rod equal to the diameter of the cold rivet at right angles to the plane of the connection, otherwise the holes shall be reamed. Likewise, when work is assembled, all holes which will not admit a rod ⅛ inch smaller than the nominal diameter of the cold rivet shall be reamed.

Since such variations are permitted in structural work, it is manifest that the nominal size of the rivet should be used in computations of stresses. Structural tables of safe shearing and bearing strengths of rivets are based on the nominal diameter. In all of the following problems in structural riveting, the nominal size is to be used.

The sizes of rivets commonly used in structural work are $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$, and 1 inch.

The riveting of plate girders will be discussed in Art. 56.

EXAMPLE

In Fig. 40, member *d* consists of two angles, each $2\frac{1}{2}$ by 2 by $\frac{5}{16}$ in., and carries a load of 16,000 lb. in tension. The rivets are $\frac{3}{4}$ in. in diameter in $\frac{13}{16}$-in. punched holes. The plate *b* is $\frac{9}{16}$ in. thick. Compute the unit tensile stress in the net section of the angles, the unit shearing stress in the rivets, and the unit bearing stress on the side of the rivets in the plate and in the angles.

Solution.—Tension. Since the holes are punched, the area deducted for the hole is $2 \times 0.875 \times 0.3125 = 0.547$ sq. in. Net area $= 2.62 - 0.547 = 2.073$ sq. in.

$$S_t = \frac{16{,}000}{2.073} = 7720 \text{ lb./in.}^2$$

Shear. The rivets are in double shear, so the shearing area is $4 \times 0.4418 = 1.767$ sq. in.

$$S_s = \frac{16{,}000}{1.767} = 9060 \text{ lb./in.}^2$$

Bearing. In the plate, the net area in bearing is $2 \times \frac{3}{4} \times \frac{9}{16} = 0.844$ sq. in.

$$S_c = \frac{16{,}000}{0.844} = 18{,}960 \text{ lb./in.}^2$$

In the angles, the net area in bearing is $2 \times \frac{3}{4} \times \frac{5}{8} = 0.9375$ sq. in.

$$S_c' = \frac{16{,}000}{0.9375} = 17{,}080 \text{ lb./in.}^2$$

PROBLEMS

1. In Fig. 40, member *c* is composed of two angles, each 3 by $2\frac{1}{2}$ by $\frac{5}{16}$ in. As in the example above, the plate is $\frac{9}{16}$ in. thick and the rivets are $\frac{3}{4}$ in. in diameter in $\frac{13}{16}$-in. punched holes. Compute the unit shearing stress in the rivets and the unit bearing stress on the side of the rivets in the plate and in the angles due to a load of 25,000 lb.

Ans. $S_s = 9430$ lb./in.^{2}; $S_c = 19{,}750$ lb./in.^{2}; $S_c' = 17{,}780$ lb./in.^{2}

2. In Fig. 41, the girder is a 15-in. 55-lb. I-beam and carries an end reaction of 60,000 lb. to the column through the three angles, each 5 by 5 by $\frac{1}{2}$ in. The rivets are $\frac{3}{4}$ in. in diameter. Compute the unit shearing stress in the rivets between the angles and the flange plate of the column, and the unit bearing stress on the side of the rivets in the angles. Neglecting the bending moment of the girder on the column and assuming two-thirds of the reaction to be carried by the four rivets through the web of the girder, compute the unit shearing stress in these rivets and the unit bearing stress in the web of the girder and in the connection angles.

Ans. 11,300 lb./in.^{2}; 13,300 lb./in.^{2}; 11,300 lb./in.^{2}; 20,600 lb./in.^{2}; 13,300 lb./in.^{2}.

3. The structural joint shown in Fig. 42 is a part of the frame of a steam shovel. Member a is composed of four angles, each 3 by 3 by $\frac{3}{8}$ in., and carries a load of 75,000 lb. in compression. Member b consists of two plates, each 8 by $\frac{3}{8}$ in. Member c consists of one angle 6 by 6 by $\frac{3}{8}$ in. The gusset plate d is $\frac{5}{8}$ in. thick, and rivets are $\frac{7}{8}$ in. in diameter in $1\frac{5}{16}$-in. punched holes. Compute the total stresses in members b and c. Compute the unit shearing stress in the rivets in member a, the unit bearing stress on the side of the rivets in member a, and the unit bearing stress on the side of the rivets in the gusset plate at member a. Compute the maximum unit tensile stress in member b, the unit shearing stress in the rivets in member b, the unit bearing stress on the side of the rivets in member b, and the unit bearing stress on the side of the rivets in the gusset plate at member b. Compute the unit shearing stress in the rivets in member c, the unit bearing stress on the side of the rivets in member c, and the unit bearing stress on the side of the rivets in the gusset plate at member c.

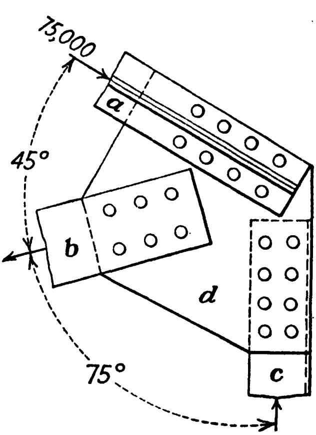


FIG. 42.

Ans. 67,230 lb.; 54,890 lb. In member a, S_s = 7790 lb./in.^{2}; S_c = 14,290 lb./in.^{2}; S_c' = 17,140 lb./in.^{2}. In member b, S_t = 14,940 lb./in.^{2}; S_s = 9320 lb./in.^{2}; S_c = 17,070 lb./in.^{2}; S_c' = 20,490 lb./in.^{2}. In member c, S_s = 11,410 lb./in.^{2}; S_c = 20,910 lb./in.^{2}; S_c' = 12,550 lb./in.^{2}.

23. Design of Structural Riveted Joints.—In the design of riveted joints in structural work, the allowable unit stresses in the plates in tension and compression, and in the rivets in shear and compression, either are given or are assumed. As a first consideration, there must be enough area of cross section of rivet or rivets to carry the load in shear. This area may be furnished by a certain number of rivets of a given size, by a smaller number of rivets of a larger size, or by a larger number of rivets of a smaller size. Difficulty of riveting the large rivets limits the size in one direction, and difficulty of punching or drilling smaller holes through a plate limits the size in the other direction.

With the size and number of rivets determined, the thickness of the plate necessary to carry the stress in compression may next be computed. If the plate as thus determined is excessively thin, a smaller number of a larger size of rivets may be used and a new computation made. If, on the other hand, the plate is excessively thick, a larger number of smaller rivets should be used.

Steel plates usually vary by sixteenths of an inch, but can be obtained on order by thirty-seconds of an inch.

In the case of a tie plate, after the thickness of the plate has been determined, the width necessary to carry the load in tension may be computed.

For end connections of girders attached to columns, the thickness of the web is known, so with an assumed size of rivet, the number of rivets necessary to carry the end reaction in compression should be computed first. Examination should then be made in shear and the number changed if necessary.

The minimum spacing of rivets is usually determined by the size of the riveter and must be large enough so that the head of a previously driven rivet does not interfere with the riveter. A minimum spacing of three times the diameter of the rivets is a common requirement.

EXAMPLE 1

Design a lap joint in a tie bar to carry a load of 42,000 lb., using allowable values as follows: S_t = 12,000 lb./in.^{2}; S_s = 9000 lb./in.^{2}; S_c = 18,000 lb./in.^{2}. Rivet holes are to be punched.

Solution.—Assume first $\frac{3}{4}$-in. rivets. The shear area required is 42,000/9000 = 4.67 sq. in. The number of rivets is 4.67/0.4418 = 10.6, so 11 rivets will be required.

In bearing, $11 \times 18{,}000 \times 0.75 \times t = 42{,}000$.

$$t = 0.283 \text{ in.}$$

It will be necessary to use a $\frac{5}{16}$-in. plate.

If there is one rivet in the first row, the width p of the plates is given by the expression,

$$(p - \tfrac{7}{8}) \times \tfrac{5}{16} \times 12{,}000 = 42{,}000.$$
$$p = 12.075 \text{ in.}$$

If there are two rivets in the first row, the width p = 12.95 in.

Another solution will now be made, using $\frac{7}{8}$-in. rivets. For $\frac{7}{8}$-in. rivets, the number of rivets is 4.67/0.6013 = 7.76, so 8 rivets will be required.

In bearing, $8 \times 18{,}000 \times 0.875 \times t = 42{,}000.$

$$t = 0.333 \text{ in.}$$

It will be necessary to use a $\frac{3}{8}$-in. plate.

If there is one rivet in the first row, the width p of the plates is given by the expression,

$$(p - 1) \times \tfrac{3}{8} \times 12{,}000 = 42{,}000.$$
$$p = 10.33 \text{ in.}$$

If there are two rivets in the first row, the width $p = 11.33$ in.

In any of these designs, plates which are thicker and narrower may be used, the only requirement of the design being that the net area of the plate in the first row of rivets shall be sufficient.

For the design using $\frac{3}{4}$-in. rivets with one rivet in the first row, plates $\frac{1}{2}$ in. thick and 7.875 in. wide could be used. This plate is slightly heavier than the one which is thinner and wider, but it might be preferable on account of being narrower.

EXAMPLE 2

A 15-in. 50-lb. I-beam used as a girder carrying a total load of 56,000 lb. uniformly distributed is to be connected to columns at each end. Design the end connections, each to consist of two angle sections, using allowable values of the unit stresses as follows: $S_s = 10{,}000$ lb./in.^{2}; in plain bearing, $S_c = 16{,}000$ lb./in.^{2}; in enclosed bearing, $S_c = 20{,}000$ lb./in.^{2}.

Solution.—Each end reaction is 28,000 lb. The thickness of the web of the I-beam is 0.55 in. Assume $\frac{3}{4}$-in. rivets and let n be the number of rivets through the web of the girder. Since this is enclosed bearing,

$$20{,}000 \times \tfrac{3}{4} \times 0.55 \times n = 28{,}000.$$
$$n = 3.4.$$

Four rivets must be used. The number of rivets required in shear will now be determined. The rivets through the web of the girder are in double shear.

$$10{,}000 \times 2 \times 0.4418 \times n = 28{,}000.$$
$$n = 3.2.$$

Four rivets will be sufficient in shear also. The clear depth of web of the I-beam is 12.468 in., so a spacing or pitch of 3 in. can be used. The connection angles will be made 12 in. long.

The rivets connecting the angles to the column are in single shear, so the number required would be

$$n' = 2 \times 3.2 = 6.4.$$

Seven rivets would be sufficient. On account of symmetry, however, it will be necessary to use 8 rivets, 4 on each side.

The thickness of the angles will now be determined. These are in plain bearing, so

$$16{,}000 \times \tfrac{3}{4} \times t \times 4 \times 2 = 28{,}000.$$
$$t = 0.292 \text{ in.}$$

The nearest commercial thickness larger than 0.292 in. is 0.3125 in., or $\frac{5}{16}$ in.

From the table of clearances required for riveting, and distance to edge of plate, the minimum width to the inside of the leg of the angle would be $1.25 + 1.25 = 2.5$ in. This, added to the thickness of the angle gives 2.8125 in. for the minimum size of the angle. The nearest commercial size above this is 3 in., so 3- by 3-in. angles could be used. The standard connection angle for this size of beam as given by the American Bridge Company is 4 by 3½ by ⅜ in., with four ⅞-in. rivets. This size permits heavier loading than assumed above. The angle with unequal legs is used so that there is no danger of interchanging the two legs.

PROBLEMS

1. In Example 1 above, make a design using 1-in. rivets.

Ans. 6 rivets; $t = \frac{7}{16}$ in.; $p = 9.125$ in.

2. Design a butt joint in a tie plate to carry a load of 24,000 lb. with the following allowable values of the unit stresses:

$S_t = 14{,}000$ lb./in.^{2}; $S_s = 9{,}000$ lb./in.^{2}; $S_c = 19{,}000$ lb./in.^{2}.

(As in Example 1, several designs are possible.)

Ans. For ⅝-in. rivets, plates 4.67 by $\frac{7}{16}$ in.; five rivets on each side, butt straps $\frac{5}{16}$ in. thick.

3. Design angle connections for the beam described in Example 2 above if ⅝-in. rivets are used, with the same allowable values.

Ans. Connection angles 3 by 3 by $\frac{5}{16}$ in., 10 in. long, with 5 rivets through the web, 10 rivets to the column.

4. A 36-in. 150-lb. I-beam used as a girder is to carry a total uniformly distributed load of 480,000 lb. Design the end connection angles, assuming ⅞-in. rivets, and using the following allowable values: for power-driven rivets in the web, $S_s = 13{,}500$ lb./in.^{2}; plain bearing $S_c = 24{,}000$ lb./in.^{2}; enclosed bearing $S_c = 30{,}000$ lb./in.^{2}; for hand-driven rivets in the outstanding legs, $S_s = 10{,}000$ lb./in.^{2}; plain bearing, $S_c = 16{,}000$ lb./in.^{2}. The thickness of the web is 0.625 in. and the clear depth of the web is 32.25 in.

Ans. Connection angles 6 by 6 by $\frac{7}{16}$ in., 30 in. long, with 15 rivets through the web, 40 rivets through the outstanding legs.

24. Boiler, Tank, and Pipe Riveting.—Boilers, tanks, and large cylindrical pipes which carry fluids under pressure are often built of plates curved to shape and fastened together by rivets. The riveted joints usually extend longitudinally and circumferentially.

In Fig. 43 are shown views of single-riveted, double-riveted, and triple-riveted lap joints. When there are two or more rows of rivets, the rivets may be staggered as in Fig. 43(*b*),

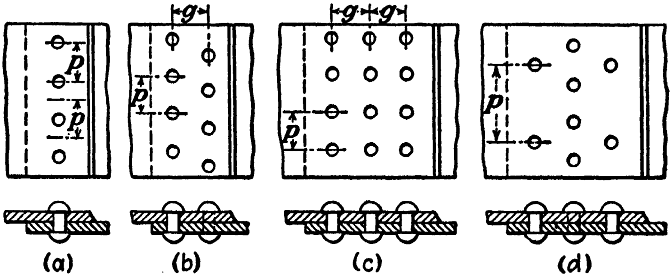


FIG. 43.

or they may be placed oppositely, as in Fig. 43(*c*). In Fig. 44 are shown views of single-riveted, double-riveted, and triple-riveted butt joints. The distance between two adjacent rivets in the same row is called the *pitch* and is denoted by *p*. The

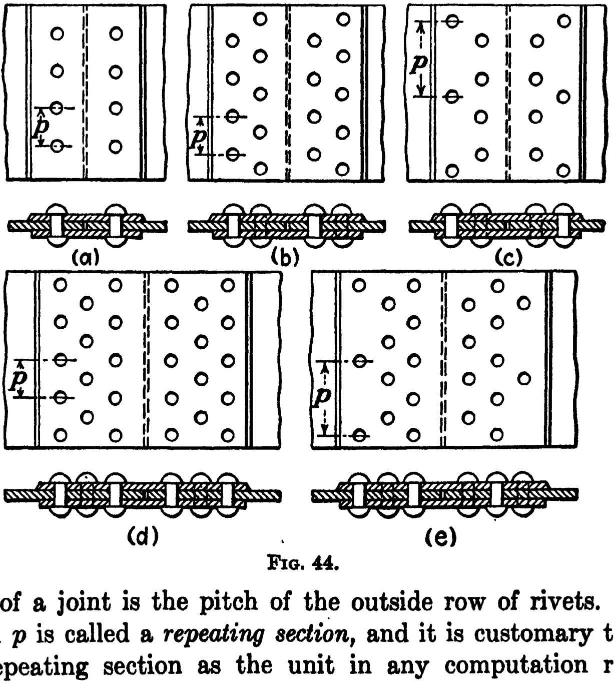


FIG. 44.

pitch of a joint is the pitch of the outside row of rivets. The length *p* is called a *repeating section*, and it is customary to use the repeating section as the unit in any computation rather than the entire length of the joint. The distance between two adjacent rows of rivets is called the *gage* and is denoted by g, as shown in Fig. 43(*b*) and 43(*c*).

It is a common requirement in specifications for boiler, tank, and pipe riveting that the minimum distance permitted between rivets shall be three times the nominal diameter of the rivet. This refers to pitch in all cases, to gage if the rivets are opposite in the rows, and to diagonal distance if the rivets are staggered.

To aid in making a tight joint, the exposed edges of the plates are caulked. This is done with a caulking tool which upsets or thickens the edge of the plate, forcing it tight against the surface of the adjoining plate. In order that proper caulking of the joints may be done, a maximum pitch of eight or ten times the thickness of the plate is all that is permitted. Plates which are to be caulked are sheared with a bevel edge. Caulking is usually done on one side only.

The sizes of rivets for boilers, tanks, and pipes are the same as those used in structural work, with the addition of some larger sizes. The sizes most commonly used are $\frac{3}{4}$, $\frac{7}{8}$, and 1 inch, but larger sizes up to $1\frac{1}{2}$ inch are in common use. Even sizes as large as $2\frac{1}{2}$ inches are used in heavy cracking stills for crude oils. As in structural work, rivet sizes vary by eighths of an inch. As ordinarily rolled, plates vary by sixteenths of an inch, but intermediate sizes may be obtained on special order.

In boilers, tanks, and pipes it is customary to require that if holes are punched, they shall be subpunched and then reamed to size. Such work is subpunched, then assembled and reamed so that holes are always concentric. If the metal is too thick to be punched, plates which are to fit together are assembled and then drilled. Rivets are usually power-driven under well-controlled conditions. On this account it is customary in such work to consider that the finished rivet completely fills the hole, so the size of the finished rivet is used in computing shearing and bearing stresses and strengths. The diameter of the actual rivet hole is used in computing the area to be deducted in obtaining the net section in tension. This method will be used in the solution of all problems in boiler, tank, and pipe riveting.

25. Strength and Efficiency of Lap Joints.—The *strength* of a joint is the total load it will hold in any of the ways in which it may possibly fail. A single-riveted or double-riveted lap joint can fail in any one of three ways, and only three.

1. *Tension in the net section of the plate in the row of rivets.* If S_t is the ultimate unit tensile strength of the plate, t its thickness, and d the diameter of the rivet hole (and the finished rivet), the ultimate strength in tension of the repeating section p in length is

$$P_t = (p - d)tS_t.$$

2. *Shear of the cross-section area of the rivet or rivets.* If n is the number of rivet areas in length p, S_s the ultimate unit shearing strength of the rivets, and $A = \pi d^2/4$ the area of the cross section of the rivet, the ultimate strength in shear of the rivets in the repeating section p in length is

$$P_s = nAS_s.$$

3. *Bearing on the sides of the rivets or rivet holes.* If m is the number of rivets in length p, and S_c the ultimate unit bearing strength of the rivets or rivet holes, the ultimate strength in bearing of the rivets in the repeating section p in length is

$$P_c = mtdS_c.$$

The expressions just given apply also to triple-riveted or quadruple-riveted lap joints if the pitch of rivets in all rows is the same. If, for either of these joints, however, the pitch in the inner row or rows is less than that in the outer rows, the joint may fail in a fourth way:

4. *Tension in the net section of the plate in the second row and shear (or bearing) of the rivet in the first row.* Let P_1 be the strength of one rivet in shear (or bearing, if it is less). The ultimate strength of the repeating section p in length in this fourth way is

$$P_{t_2} + P_1 = (p - 2d)tS_t + AS_s \text{ (or } tdS_c).$$

The strength of length p of the unriveted plate in tension is

$$P = ptS_t.$$

The *efficiency* of a joint is the ratio of the least strength of the joint to the strength of the unriveted plate in tension.

The following table gives the ultimate unit stresses for boiler steel as given in the Boiler Code of the American Society of

Mechanical Engineers, and also for tank and pipe steel as given by the Chicago Bridge and Iron Works.

ULTIMATE UNIT STRESSES
(Pounds per Square Inch)

Unit stress	Boiler steel (A. S. M. E.)	Tank steel (C. B. & I. W.)
S_t	55,000	60,000
S_s	44,000	45,000
S_c	95,000	90,000

In recent years, a 3 per cent nickel steel with values averaging 30 per cent higher than those given in the American Society of Mechanical Engineers Boiler Code, has been used in locomotives on the Canadian Pacific Railway. The use of the stronger material permitted the increase of boiler pressures from 200 pounds per square inch up to 250 pounds per square inch without any increase in the thickness of the boiler plate.

EXAMPLE 1

A double-riveted lap joint for ¼-in. plates has ½-in. rivets with a pitch of 2⅛ in. Compute the three strengths and the efficiency of the joint, using the values given by the Boiler Code of the American Society of Mechanical Engineers.

Solution.—The rivet holes are ¹⁄₁₆ in. larger than the undriven rivet, so $d = \frac{9}{16}$ in.

$$P_t = (2.125 - 0.5625) \times 0.25 \times 55{,}000 = 21{,}480 \text{ lb.}$$
$$P_s = 2 \times 0.2485 \times 44{,}000 = 21{,}870 \text{ lb.}$$
$$P_c = 2 \times 0.5625 \times 0.25 \times 95{,}000 = 26{,}720 \text{ lb.}$$

Strength P_t is the smallest, and therefore governs.

$$P = 2.125 \times 0.25 \times 55{,}000 = 29{,}220 \text{ lb.}$$
$$\text{Efficiency} = \frac{21{,}480}{29{,}220} = 0.735, \text{ or } 73.5 \text{ per cent.}$$

EXAMPLE 2

A quadruple-riveted lap joint for ⅜-in. plates has ¾-in. rivets with a pitch of 3.55 in. in the two inner rows and a pitch of 7.1 in. in the two outer rows. Compute the four strengths and the efficiency of the joint, using the values given by the Chicago Bridge and Iron Works.

Solution.—The length of the repeating section is 7.1 in. and there are six rivets in this length. The diameter of the rivet holes is ¹³⁄₁₆ = 0.8125 in.

$$P_t = (7.1 - 0.8125) \times 0.375 \times 60{,}000 = 141{,}470 \text{ lb.}$$
$$P_s = 6 \times 0.5185 \times 45{,}000 = 140{,}000 \text{ lb.}$$
$$P_c = 6 \times 0.8125 \times 0.375 \times 90{,}000 = 164{,}530 \text{ lb.}$$

The strength in shear is less than the strength in bearing, and

$$P_1 = \frac{140{,}000}{6} = 23{,}330 \text{ lb.}$$
$$P_{t_2} + P_1 = (7.1 - 1.625) \times 0.375 \times 60{,}000 + 23{,}330 = 146{,}520 \text{ lb.}$$

Strength P_s is the smallest of the four.

$$P = 7.1 \times 0.375 \times 60{,}000 = 159{,}750 \text{ lb.}$$
$$\text{Efficiency} = \frac{140{,}000}{159{,}750} = 0.876, \text{ or } 87.6 \text{ per cent.}$$

PROBLEMS

1. A single-riveted lap joint for $\frac{3}{16}$-in. plates has $\frac{5}{8}$-in. rivets with a pitch of 2 in. Compute the efficiency of the joint, using the values of the American Society of Mechanical Engineers. *Ans.* 59.4 per cent.

2. A double-riveted lap joint for $\frac{7}{16}$-in. plates has 1-in. rivets with a pitch of 3.5 in. Compute the efficiency of the joint, using the values of the American Society of Mechanical Engineers. *Ans.* 69.6 per cent.

3. A triple-riveted lap joint for $\frac{5}{16}$-in. plates has $\frac{5}{8}$-in. rivets, with a pitch of 2 in. in the inner row and a pitch of 4 in. in the outer rows. Compute the efficiency of the joint, using the values of the Chicago Bridge and Iron Works. *Ans.* 82.8 per cent.

4. Solve Problem 3 if the pitch is 2 in. in all rows. *Ans.* 65.6 per cent.

5. A quadruple-riveted lap joint for $\frac{1}{2}$-in. plates has $\frac{7}{8}$-in. rivets with a pitch of 5 in. in all rows. Compute the efficiency of the joint, using the values of the Chicago Bridge and Iron Works. *Ans.* 81.3 per cent.

26. Strength and Efficiency of Butt Joints.

—If the two straps of a butt joint are the same width and the pitch of the rivets is the same in all of the rows, the joint can fail in any one of the same three principal ways as a lap joint.

1. *Tension in the net section of the main plate in the first row of rivets.*

$$P_t = (p - d)tS_t.$$

2. *Double shear of the rivet areas.*

$$P_s = nAS_s.$$

3. *Bearing on the sides of the rivets or rivet holes in the main plate.*[1]

---

[1] Contrary to the procedure in structural riveting, it is not customary in boiler, tank, or pipe riveting to allow higher values in enclosed bearing than in plain bearing. The same unit stress in bearing is used in the main plates as in the butt straps.

$$P_c = mtdS_c.$$

If the pitch is greater in the first row than in the second row, there are two other possible ways in which the joint may fail.

4. *Tension in the main plates in the second row and shear (or bearing) of the rivet in the first row.*

$$P_{t_2} + P_1 = (p - 2d)tS_t + 2AS_s \text{ (or } tdS_c\text{)}.$$

5. *Tension in the net section of the butt straps in the inner row of rivets.*

$$P_t' = 2(p - nd)t'S_t.$$

If one of the butt straps is wider than the other, some of the rivets are in double shear and some are in single shear. In (3), the bearing is in the main plate for the rivets which are in double shear, but in the butt strap for the rivet in single shear. In (4), the bearing is on the side of the rivet in the butt strap.

In case there are two or more extra rows of rivets through the wide butt strap, there are still other ways in which failure may occur, as by tension in the third row and shear (or compression) of the rivets in both the first and second rows. Each type of multiple-riveted joint must be examined in all of the possible ways in which it may fail.

EXAMPLE 1

A double-riveted butt joint for 1-in. plates has ⅝-in. butt straps, both of the same width, and 1¼-in. rivets with a pitch of 7.6 in. in the first row and one of 3.8 in. in the second row. Compute the strengths and the efficiency of the joint, using the values of the American Society of Mechanical Engineers.

Solution:

$$\begin{aligned} P_{t_1} &= (7.6 - 1.3125) \times 1 \times 55{,}000 = 345{,}800 \text{ lb.} \\ P_s &= 6 \times 1.353 \times 44{,}000 = 357{,}200 \text{ lb.} \\ P_c &= 3 \times 1 \times 1.3125 \times 95{,}000 = 374{,}000 \text{ lb.} \end{aligned}$$

The strength in shear is less than that in bearing, and the strength of one rivet in shear is $P_1 = 357{,}200/3 = 119{,}000$ lb.

$$\begin{aligned} P_{t_2} + P_1 &= (7.6 - 2.625) \times 1 \times 55{,}000 + 119{,}000 = 392{,}600 \text{ lb.} \\ P_t' &= (7.6 - 2.625) \times 1.25 \times 55{,}000 = 342{,}000 \text{ lb.} \end{aligned}$$

The value of P_t' is the smallest of the five.

$$P = 7.6 \times 1 \times 55{,}000 = 418{,}000 \text{ lb.}$$

$$\text{Efficiency} = \frac{342{,}000}{418{,}000} = 0.818, \text{ or } 81.8 \text{ per cent.}$$

EXAMPLE 2

A triple-riveted butt joint has ½-in. main plates, ⅜-in. butt straps, and ¾-in. rivets with a pitch of 6 in. in the first row and one of 3 in. in the other two. The rivets in the first row extend through only one butt strap and the main plate, while those in the second and third rows extend through both butt straps and the main plate. Compute the five strengths and the efficiency of the joint, using the values of the Chicago Bridge and Iron Works.

Solution:

$$P_{t_1} = (6 - 0.8125) \times 0.5 \times 60{,}000 = 155{,}600 \text{ lb.}$$
$$P_s = 9 \times 0.5185 \times 45{,}000 = 210{,}000 \text{ lb.}$$
$$P_c = (4 \times 0.5 \times 0.8125 + 0.375 \times 0.8125) \times 90{,}000 = 173{,}700 \text{ lb.}$$

The strength of one rivet area in shear is 210,000/9 = 23,330 lb. The strength of one rivet in bearing in the butt strap is 0.375 × 0.8125 × 90,000 = 27,420 lb. The value of P_1 is therefore 23,330 lb.

$$P_{t_2} + P_1 = (6 - 1.625) \times 0.5 \times 60{,}000 + 23{,}330 = 154{,}580 \text{ lb.}$$
$$P_t' = (6 - 1.625) \times 0.75 \times 60{,}000 = 196{,}875 \text{ lb.}$$

The value of 154,580 lb. is the least of the five.

$$P = 6 \times 0.5 \times 60{,}000 = 180{,}000 \text{ lb.}$$
$$\text{Efficiency} = \frac{154{,}580}{180{,}000} = 0.859, \text{ or } 85.9 \text{ per cent.}$$

PROBLEMS

1. A single-riveted butt joint has ½-in. main plates, 5/16-in. butt straps, and ⅞-in. rivets with a pitch of 2⅝ in. Compute the efficiency of the joint, using the values of the American Society of Mechanical Engineers.

Ans. 61.6 per cent.

2. A double-riveted butt joint has 11/16-in. main plates, 7/16-in. butt straps, both of the same width, and ⅞-in. rivets with a pitch of 6 in. in the outer row and one of 3 in. in the inner row. Compute the efficiency of the joint, using the values of the American Society of Mechanical Engineers.

Ans. 80.3 per cent.

3. Solve Problem 2 if the pitch is 3 in. in both rows.

Ans. 68.7 per cent.

4. A triple-riveted butt joint has ¾-in. main plates, ½-in. butt straps, and 1⅛-in. rivets with a pitch of 10 in. in the outer row and one of 5 in. in the other two. The rivets in the first row extend through only one butt strap and the main plate, while those in the second and third rows extend through both butt straps and the main plate. Compute the efficiency of the joint, using the values of the American Society of Mechanical Engineers.

Ans. 88.0 per cent.

5. A quadruple-riveted butt joint has ½-in. main plates, ⅜-in. butt straps, and ⅞-in. rivets. The rivets in the first and second rows extend through only one butt strap and the main plate, while those in the third and fourth rows extend through both butt straps and the main plate. The pitch in the first row is 15.4 in., that in the second row is 7.7 in., and that in the third and fourth rows is 3.85 in. Compute the efficiency of the joint, using the values of the Chicago Bridge and Iron Works.

Ans. 93.2 per cent.

27. Stresses across Longitudinal Joints in Thin Cylinders.—Boilers, pipes, and tanks, as ordinarily constructed, are thin cylinders in which the tensile stresses in the plate may be considered to be uniform throughout the thickness of the plate at any section.

In a thin cylinder which is subjected to internal fluid pressure, the pressure is normal to the surface at all points. These normal pressures develop tensile stresses at all sections in the walls of the cylinder. Let Fig. 45 represent one-half of a ring cut from such a cylinder, and let its length be unity in the direction of the axis of the cylinder. Let S_t be the unit tensile stress in the wall of the cylinder, t the thickness of the wall, r the radius of the cylinder, and R the internal pressure on unit area. At A in Fig. 45, the pressure upon the differential area one unit in length and $rd\theta$ in width is $Rrd\theta$. The horizontal component of this pressure is $Rr\cos\theta d\theta$. The total horizontal pressure to the right is given by the expression, $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} Rr\cos\theta d\theta$. This pressure to the right is balanced by the tensile forces in the metal acting to the left, $2S_t t$.

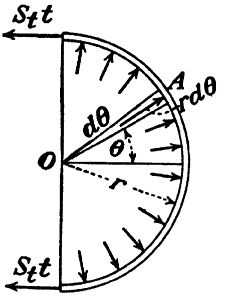


FIG. 45.

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} Rr\cos\theta d\theta = 2S_t t.$$

$$Rr\sin\theta\Big]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} = 2S_t t.$$

$$Rr = S_t t.$$

This is the relation between the internal pressure and the stress in a circumferential direction, tending to open a longitudinal crack.

If there are longitudinal riveted joints in the cylinder, the *effective net thickness* of the plate is the product of the actual thickness of the plate and the efficiency of the joint. For example, if the plates of a boiler are ½ inch thick and a longitudinal joint has an efficiency of 80 per cent, the effective net thickness of the plate is $0.5 \times 0.80 = 0.4$ inch. That is, a plate 0.4 inch thick with no joint would be just as strong as the ½-inch plate with a joint. For cylinders with longitudinal riveted joints, in terms of the actual thickness of the plates,

$$Rr = S_t t \times \text{eff.}$$

In tanks and pipes containing liquids and open to the air at the top, the pressure at any point is equal to the weight of a column of the liquid of the same height as the vertical distance from the point to the surface of the liquid. This vertical distance is called the *head.* If the head in feet is h, the pressure in pounds per square inch for water which weighs 62.5 pounds per cubic foot is

$$R = \frac{62.5h}{144} = 0.434h.$$

A head of 100 feet of water causes a pressure of 43.4 pounds per square inch.

Oil is lighter than water, and for an average weight of 58 pounds per cubic foot, the pressure per square inch is $R = 0.403h$. For tar which averages 75 pounds per cubic foot, the pressure per square inch is $R = 0.521h$.

EXAMPLE 1

A water pipe 6 ft. in diameter is made of ½-in. plates, and the longitudinal joints are triple-riveted lap joints, with ⅞-in. rivets. The pitch is 2¾ in. in the inner row and 5½ in. in the outer rows. Compute the factor of safety when the head of water on the pipe is 300 ft. Use the values of the Chicago Bridge and Iron Works.

Solution:

$$P_{t_1} = (5.5 - 0.9375) \times 0.5 \times 60{,}000 = 136{,}880 \text{ lb.}$$
$$P_s = 4 \times 0.6903 \times 45{,}000 = 124{,}250 \text{ lb.}$$
$$P_c = 4 \times 0.5 \times 0.9375 \times 90{,}000 = 168{,}750 \text{ lb.}$$

$$P_{t_2} + P_1 = (5.5 - 1.875) \times 0.5 \times 60{,}000 + 31{,}060 = 139{,}810 \text{ lb.}$$
$$P = 5.5 \times 0.5 \times 60{,}000 = 165{,}000 \text{ lb.}$$
$$\text{Efficiency} = \frac{124{,}250}{165{,}000} = 0.753.$$

The effective net thickness = 0.5 × 0.753 = 0.3765 in.

$$R = 0.434 \times 300 = 130.2 \text{ lb./in.}^2.$$
$$130.2 \times 36 = S_t \times 0.3765.$$
$$S_t = 12{,}420 \text{ lb./in.}^2.$$
$$\text{Factor of safety} = \frac{60{,}000}{12{,}420} = 4.83.$$

EXAMPLE 2

A boiler 4 ft. in diameter is made of 1¼-in. plates and has double-riveted longitudinal butt joints with 1⅜-in. rivets which have a pitch of 5 in. in both rows. The butt straps are ¾ in. thick and both are the same width. Using the values of the American Society of Mechanical Engineers, compute the safe steam pressure with a factor of safety of 4.

Solution:

$$P_t = (5 - 1.4375) \times 1.25 \times 55{,}000 = 244{,}920 \text{ lb.}$$
$$P_s = 4 \times 1.623 \times 44{,}000 = 285{,}650 \text{ lb.}$$
$$P_c = 2 \times 1.25 \times 1.4375 \times 95{,}000 = 341{,}410 \text{ lb.}$$
$$P = 5 \times 1.25 \times 55{,}000 = 343{,}750 \text{ lb.}$$
$$\text{Efficiency} = \frac{244{,}920}{343{,}750} = 0.712.$$
$$R \times 24 = 13{,}750 \times 1.25 \times 0.712.$$
$$R = 510 \text{ lb./in.}^2.$$

PROBLEMS

1. An open cylindrical water tank with the axis vertical is 25 ft. in diameter and 40 ft. high. It is made of ¼-in. plates with double-riveted lap joints. Rivets are ½ in. in diameter with a pitch of 2 in. Compute the factor of safety when the tank is full of water, using the values of the Chicago Bridge and Iron Works. *Ans.* 4.14.

2. A cylindrical boiler is made of ⅜-in. plates and has longitudinal single-riveted lap joints with ⅞-in. rivets and a pitch of 2¼ in. Using the values of the American Society of Mechanical Engineers and a factor of safety of 5, compute the maximum diameter of boiler for a steam pressure of 120 lb./in.². *Ans.* 40 in.

3. A boiler 5 ft. in diameter is made of 1⅛-in. plates. The longitudinal joints are triple-riveted butt joints and the rivets are 1½ in. in diameter. The pitch in the outer row is 10 in. and that in the inner rows is 5 in. The butt straps are 27/32 in. thick. The rivets in the first row extend through only one butt strap and the main plate, while those in the second and third rows extend through both butt straps and the main plate. Using the values of the American Society of Mechanical Engineers and a factor of safety of 4.5, compute the safe steam pressure. *Ans.* 377 lb./in.^{2}.

4. An oil tank 117 ft. in diameter has the bottom ring made of ½-in. plates. The vertical joints are quadruple-riveted lap joints with ¾-in. rivets and a pitch of 3.53 in. in all rows. Using the values of the Chicago Bridge and Iron Works, determine the safe height of the tank with a factor of safety of 3. *Ans.* 27.2 ft.

5. If the vertical joints of the surge tank of a water-power pipe line are to have an efficiency of 90 per cent and the tank is to be 55 ft. in diameter and 130 ft. high, compute the thickness of the bottom plate with a factor of safety of 4. Use the values of the Chicago Bridge and Iron Works.

Ans. 1⅜ in.

28. Stresses across Circumferential Joints in Thin Cylinders and in Spheres.

—To determine the stress in the longitudinal direction in a cylinder tending to open a circumferential crack, consider the part of the cylinder shown in Fig. 46 as the free body. The total longitudinal pressure of the fluid with a unit pressure of R pounds per square inch is $\pi r^2 R$. The balancing longitudinal stress in the metal is $2\pi r t S_t$. Since these are equal,

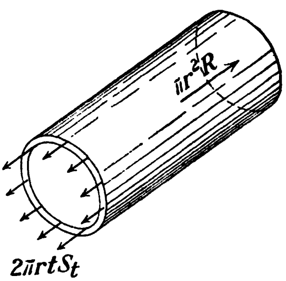


FIG. 46.

$$\pi r^2 R = 2\pi r t S_t.$$
$$R r = 2 S_t t.$$

It will be seen from this expression that the stress tending to open a circumferential crack in a cylinder is only one-half as much as that tending to open a longitudinal crack, so the required efficiency of a circumferential joint in a boiler or pipe is only one-half that of the longitudinal joint. Even though a multiple-riveted butt joint of high efficiency is required for the longitudinal joints, a single-riveted or double-riveted lap joint will usually be sufficient for the circumferential joints. In the case of tanks open to the air at the top and with the axis vertical, the only stresses in the circumferential joints are those caused by the weight of the tank above and by the wind. The chief requirement is that the joint shall be tight enough to hold the contents of the tank.

The expression $Rr = 2S_t t$ applies also to the stress in a sphere caused by internal fluid pressure, as will be seen by considering the pressure upon the projected area of the hemisphere, which is the same as that upon the flat end of a closed cylinder.

PROBLEMS

1. Prove that for the boiler referred to in Problem 3, Art. 27, a double-riveted lap joint with a pitch of 5 in. would be strong enough for the circumferential joints, but a single-riveted lap joint with the same pitch would not.

2. A spherical gas container 45 ft. in diameter is to hold gas at a pressure of 40 lb./in.^{2}. If it is to have double-riveted lap joints with an efficiency of 72 per cent, what must be the thickness of the plates for an allowable tensile stress of 15,000 lb./in.^{2}? *Ans.* ½ in.

3. A spherical gas container is 60 ft. in diameter and is made of ⅝-in. plates. The joints are triple-riveted lap joints, with ⅞-in. rivets and a pitch of 3.16 in. in all rows. Using the values of the Chicago Bridge and Iron Works, compute the safe internal pressure with a factor of safety of 4. *Ans.* 36.6 lb./in.^{2}.

29. Design of Joints of Maximum Efficiency.—A joint of any given type has its maximum efficiency with the minimum amount of material when it has equal strengths in all of the possible ways in which it may fail. For the simple joints which may fail in only the three principal ways, that is, by tension in the main plate, by shear of the rivets, or by bearing on the side of the rivets, a simple relation between t, d, and p may be obtained.

As before, let m be the number of rivets in bearing and n the number of rivet areas in shear. The strength of the repeating section p in shear is equated to the strength in bearing, and also the strength in tension is equated to that in bearing. In terms of the diameter d of the finished rivet, the strength of one area of the rivet in shear is $\pi d^2 S_s/4$.

$$P_s = P_c.$$

$$\frac{n\pi d^2 S_s}{4} = mtdS_c.$$

$$d = \frac{4}{\pi}\frac{m}{n}\frac{S_c}{S_s}t.$$

$$P_t = P_c.$$

$$(p - d)tS_t = mtdS_c.$$

$$p = \left(m\frac{S_c}{S_t} + 1\right)d.$$

In terms of t,

$$p = \left(m\frac{S_c}{S_t} + 1\right)\left(\frac{4}{\pi}\frac{m}{n}\frac{S_c}{S_s}\right)t.$$

$$\text{Efficiency} = \frac{p - d}{p}$$

$$= \frac{\left(m\frac{S_c}{S_t} + 1\right)d - d}{\left(m\frac{S_c}{S_t} + 1\right)d}$$

$$= \frac{mS_c}{mS_c + S_t}.$$

In the table below, the computed values of d in terms of t, of p in terms of t, and of the efficiencies for several of the simpler types of joints are given for both the values of the Boiler Code of the American Society of Mechanical Engineers and the values used by the Chicago Bridge and Iron Works. All rows have the same number of rivets. If values other than these are used and with different ratios of the allowable stresses in tension, shear, and bearing, a corresponding table should be computed.

Type of joint	A. S. M. E.			C. B. & I. W.		
	d	p	Eff.	d	p	Eff.
$L1$	$2.75t$	$7.5\ t$	0.633	$2.54t$	$6.25t$	0.60
$L2$	$2.75t$	$12.25t$	0.775	$2.54t$	$10.16t$	0.75
$L3$	$2.75t$	$17.0\ t$	0.838	$2.54t$	$14.0\ t$	0.82
$B1$	$1.38t$	$3.75t$	0.633	$1.27t$	$3.13t$	0.60
$B2$	$1.38t$	$6.13t$	0.775	$1.27t$	$5.08t$	0.75
$B3$	$1.38t$	$8.50t$	0.838	$1.27t$	$7.0\ t$	0.82

Commercial sizes of plates and rivets must of course be used. In general, the nearest commercial size of plate thicker than the computed size will be used. The efficiency of a joint will be lowered by changing the size of the rivet either way from the computed size. If a larger size is used, the efficiency in tension will be lowered. If a smaller size is used, the efficiencies in compression and in shear will be lowered. Since the size d computed is the diameter of the finished rivet and therefore of the rivet hole, the value of d selected to be used must be the ***nearest odd sixteenth*** to the value of d as computed, and the original or nominal diameter of the rivet will be the even sixteenth of an inch smaller. The pitch is ordinarily used as computed, but may be changed to the nearest eighth of an inch. Standard rules for riveting will sometimes require further modification in the design, which in general will still further reduce the efficiency.

EXAMPLE 1

Using the values of the Chicago Bridge and Iron Works and a factor of safety of 4, design a double-riveted butt joint of maximum efficiency for a water pipe 10 ft. in diameter under a head of 320 ft.

Solution.—From the table on p. 63, the maximum efficiency for this joint is 0.75.

$$\begin{aligned} Rr &= S_t t \times \text{eff.} \\ 0.434 \times 320 \times 60 &= 15{,}000 \times t \times 0.75. \\ t &= 0.74 \text{ in.} \\ d &= 1.27t = 0.94 \text{ in.} \\ p &= 5.08t = 3.759 \text{ in.} \end{aligned}$$

The commercial sizes to be used will be as follows: $t = \frac{3}{4}$ in.; $t' = \frac{7}{16}$ in.; $d = 0.9375$ in. ($\frac{7}{8}$-in. rivet).

EXAMPLE 2

Using the ultimate strengths of nickel-steel boiler plate and rivets as follows, design a triple-riveted lap joint of maximum efficiency for a boiler 5 ft. in diameter to carry a steam pressure of 250 lb./in.² with a factor of safety of 4. $S_t = 80{,}000$ lb./in.²; $S_s = 55{,}000$ lb./in.²; $S_c = 115{,}000$ lb./in.².

Solution.—The number of rivets in a repeating section is 3, so $m = 3$ and $n = 3$.

$$\begin{aligned} \text{Efficiency} &= \frac{3 \times 115{,}000}{3 \times 115{,}000 + 80{,}000} = 0.812. \\ 250 \times 30 &= 20{,}000 \times t \times 0.812. \\ t &= 0.462 \text{ in.} \\ d &= \frac{4}{\pi}\cdot\frac{m}{n}\frac{S_c}{S_s}t = 1.23 \text{ in.} \\ p &= \left(m\frac{S_c}{S_t} + 1\right)d = 6.55 \text{ in.} \end{aligned}$$

The commercial values used will be as follows: $t = \frac{1}{2}$ in.; $d = 1.1875$ in. ($1\frac{1}{8}$-in. rivet).

PROBLEMS

1. Using the values of the American Society of Mechanical Engineers with a factor of safety of 4.5, design a double-riveted lap joint for a boiler 40 in. in diameter with a steam pressure of 135 lb./in.^{2}.

Ans. Computed values: $t = 0.285$ in.; $d = 0.784$ in.; $p = 3.49$ in. Use 5/16-in. plates, and 3/4-in. rivets in 13/16-in. holes.

2. Using the values of the Chicago Bridge and Iron Works with a factor of safety of 5, design a single-riveted butt joint of maximum efficiency for a water tank 40 ft. in diameter and 50 ft. high.

Ans. Computed values: $t = 0.723$ in.; $d = 0.919$ in.; $p = 2.26$ in. Use $t = 3/4$ in.; $t' = 7/16$ in.; $d = 0.9375$ in. (7/8-in. rivet).

3. Using the values of the Chicago Bridge and Iron Works with a factor of safety of 4, design a triple-riveted butt joint of maximum efficiency for a pipe 6 ft. in diameter under a head of 820 ft. of water.

Ans. Computed values: $t = 1.042$ in.; $d = 1.323$ in.; $p = 7.291$ in. Use $t = 1\frac{1}{16}$ in.; $t' = 9/16$ in.; $d = 1.3125$ in. ($1\frac{1}{4}$-in. rivet).

4. Using ultimate $S_t = 70{,}000$ lb./in.^{2}, $S_s = 55{,}000$ lb./in.^{2}, $S_c = 110{,}000$ lb./in.^{2}, and a factor of safety of 4, design a double-riveted butt joint of maximum efficiency for a tank 100 ft. in diameter and 40 ft. high to hold tar.

Ans. Computed values: $t = 0.94$ in.; $d = 1.20$ in.; $p = 4.97$ in. Use $t = 1$ in.; $t' = 9/16$ in.; $d = 1.1875$ in. ($1\frac{1}{8}$-in. rivet).

30. Fillet-welded Joints.—Welding may be defined as the joining of two pieces of metal by means of fusion. The two surfaces to be joined are heated by means of an electric arc or an oxyacetylene torch, and material from the fused welding rod is deposited between them to form the union. The welding rod should always be of the same composition as the plates to be welded. In the fillet weld, such as is shown in Fig. 47, the fused metal is deposited in a triangular corner or fillet between the two pieces to be joined. In this case the fillet is deposited at the edge or side of the upper plate, and the weld is called a *side* fillet weld.

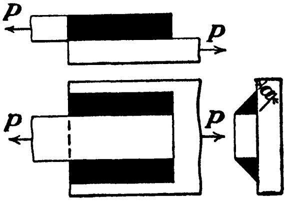


FIG. 47.

If tensile forces are applied to the joint as shown, the metal in the fillet is subjected to shearing forces. The minimum section in the fillet is along the diagonal distance a, called the *throat* of the weld, which is 0.707 times the base of the weld for a 45° fillet.

The American Welding Society gives the following allowable loads per linear inch of 45° fillet, for different sizes of fillets:

¼-in. base of weld	2000 lb./lin. in.
5/16-in. base of weld	2500 lb./lin. in.
⅜-in. base of weld	3000 lb./lin. in.
7/16-in. base of weld	3500 lb./lin. in.
½-in. base of weld	4000 lb./lin. in.

These values are at the rate of 11,300 pounds per square inch allowable shearing stress in the throat of the weld.

The weld shown in Fig. 48 is called an *end* fillet weld, and the material in the fillet is stressed both in tension and in shear. It has been found from tests by the photoelastic method that a 30°–60° fillet with the long base in shear gives a much better distribution of the stress than the ordinary 45° fillet. Such welds should be used only as auxiliary welds, and only if it is not possible to develop the required strength by means of the side welds.

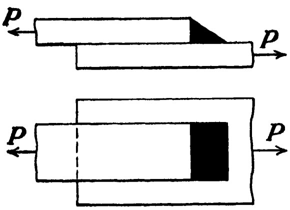


FIG. 48.

If an angle section is to be welded to a plate by means of side fillet welds, and the load in the angle is to act along the centroidal axis of the angle, it is necessary to have a longer weld along the heel of the angle than along the toe, as shown in Fig. 49. Let p_1 be the stress per linear inch in the weld. With the angle as the free body, the equation of moments about any point at the lower edge of the angle gives,

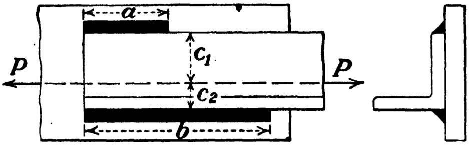


FIG. 49.

$$p_1 a(c_1 + c_2) = Pc_2.$$

$$a = \frac{Pc_2}{p_1(c_1 + c_2)}.$$

Similarly,

$$b = \frac{Pc_1}{p_1(c_1 + c_2)}.$$

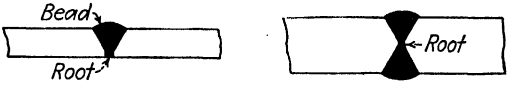
PROBLEMS

1. If the narrow plate shown in Fig. 47 is 6 by $\frac{1}{2}$ in. and is to develop an allowable stress in tension of 18,000 lb./in.^{2}, what must be the length of lap of the two plates for $\frac{1}{2}$-in. side welds? *Ans.* $6\frac{3}{4}$ in.

2. A 6-in. 8.2-lb. channel has a 7.5-in. lap on the connecting plate at its end. If it is to develop an allowable stress of 18,000 lb./in.^{2} in tension, what size of side fillet weld should be used? *Ans.* $\frac{3}{8}$-in.

3. A 6- by 4- by $\frac{5}{8}$-in. angle is to have its 6-in. leg welded to a plate by means of side fillet welds. If $\frac{7}{16}$-in. welds are to be used and the allowable tensile stress in the angle is to be 16,000 lb./in.^{2}, what lengths a and b are required? *Ans.* 9.06 in.; 17.7 in.

31. Butt-welded Joints.—In a butt-welded joint, the plates are butted together and their edges welded. Plates from $\frac{1}{16}$ to $\frac{1}{8}$ inch in thickness can be butt-welded very easily with square edges. Plates between $\frac{1}{8}$ and $\frac{1}{2}$ inch in thickness are usually beveled to form a single-V weld, as shown in Fig. 50(a). Plates thicker than $\frac{1}{2}$ inch are usually beveled on both sides to form a double-V weld, as shown in Fig. 50(b). If in either case the root of the weld is not completely filled with fused material, very large localized stresses may be caused. The excess material above the V may also be an element of weakness and cause large localized stresses in the plate at the edge of the weld. For the best results the bead should be only slightly rounded and should meet the straight surface of the plate smoothly. On account of the change of properties of the metal which has been fused, it is sometimes necessary to anneal the plates after welding.



(a) (b)

FIG. 50.

The butt weld is used extensively in steel pipe lines for gas, oil, and steam, for pressure vessels such as gas spheres, oil-cracking stills and retorts, and for open water and oil tanks. In many such tanks the floors and roofs are welded and the sides riveted, but in recent years many tanks have been erected with welded joints throughout.

The American Welding Society gives the following allowable unit stresses for butt joints in tension or compression, either direct or in bending:

Tension on section through throat of weld........ 13,000 lb./in.^{2}.
Compression on section through throat of weld... 18,000 lb./in.^{2}.

Problems involving plates with butt joints are identical with those involving plain plates, with the exception of the lower allowable value in tension.

PROBLEM

1. A steel pipe 3 ft. in diameter with metal ¼ in. thick has longitudinal-welded butt joints. Compute the safe internal pressure.

Ans. 180.6 lb./in.^{2}.

CHAPTER IV

TORSION OF SHAFTS

32. Shearing Stresses in a Circular Shaft.—As explained in Art. 17, a circular shaft which is twisted about its own axis by a torque T at each end is subjected to pure shearing stress. Let Fig. 51 represent such a shaft twisted in a counterclockwise direction by the couple $T_1 = 2Pr$ at the right end and in the opposite direction by the balancing couple $T_2 = 2Pr$ at the left end. Let Fig. 52 represent a cross section of the shaft, as that at A in Fig. 51, and let S_s be the unit shearing stress at the outer fiber.

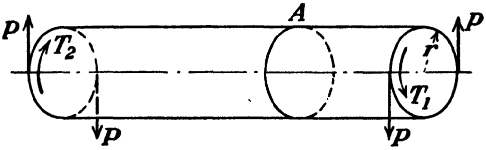


FIG. 51.

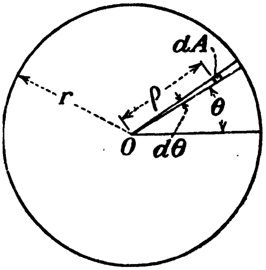


FIG. 52.

$\dfrac{S_s}{r}$ is the unit shearing stress at unit distance from the axis.

$\dfrac{S_s\rho}{r}$ is the unit shearing stress at the distance ρ from the axis.

$\dfrac{S_s\rho dA}{r}$ is the shearing force on the differential area dA.

$\dfrac{S_s\rho^2 dA}{r}$ is the moment about O of the force on the area dA.

$\dfrac{S_s}{r}\int\rho^2 dA$ is the total moment about O of all of the internal shearing forces.

The quantity $\int\rho^2 dA$ is the polar moment of inertia of the cross section and may be represented by J.

The polar moment of inertia of a circle is $J = \frac{1}{2}\pi r^4$.

$\frac{S_s J}{r}$ is the internal moment or resisting torque.

Since this internal torque holds in equilibrium the external torque T,

$$T = \frac{S_s J}{r}.$$

$$S_s = \frac{Tr}{J}.$$

The unit stress at any point other than the outer fiber is proportional to its distance from the center of the shaft and is given by the expression,

$$S_s = \frac{T\rho}{J}.$$

PROBLEMS

1. A solid steel shaft 3 in. in diameter is twisted by a force of 1000 lb. at the end of a lever 4 ft. long. Compute the maximum shearing stress. *Ans.* 9050 lb./in.^{2}.

2. If allowable S_s in a steel shaft is 10,000 lb./in.^{2}, what torque T is safe for a shaft 16 in. in diameter? *Ans.* 670,200 ft.-lbs.

3. If safe S_s = 9000 lb./in.^{2}, what must be the diameter of a solid circular shaft subjected to a torque of 120 in.-lb.? *Ans.* 0.408 in.

4. A steel oil pipe is 8.625 in. outside diameter and 7.981 in. inside diameter. In tightening up the screwed joints of the pipe, six men exert a pressure of 150 lb. each on a pipe wrench at an average distance of 10 ft. from the center of the pipe. Compute the maximum shearing stress. *Ans.* 3230 lb./in.^{2}.

5. If the inside diameter of a shaft is to be $\frac{8}{10}$ of the outside diameter, what must be the size of a shaft to transmit a torque of 84,000 in.-lb. with an allowable shearing stress of 7500 lb./in.^{2}? *Ans.* 4.6 in.

6. If the shaft described in Problem 5 is steel, how much weight is saved per foot by using it instead of a solid shaft? *Ans.* 19.5 lb.

33. Shearing Strain in a Circular Shaft.—When a cylindrical shaft is subjected to a torque, any line which was originally an element of the cylinder is twisted about the axis and takes a helical shape, as shown in Fig. 53. In this figure, OQ is the axis of the cylinder and AB is the original position of an element. If the radius AQ remains fixed in position as the torque $2Pr$ is applied to each end, the radius OB will twist through the angle θ and will take the position OB', and the element AB will take the helical shape AB'. The arc BB' is therefore the total shearing deformation at the outer surface of the shaft in length l, and in amount is equal to $r\theta$. The unit shearing deformation, δ_s is given by the expression

$$\delta_s = \frac{r\theta}{l}.$$

Since

$$S_s = E_s\delta_s,$$

$$S_s = \frac{E_s r\theta}{l}.$$

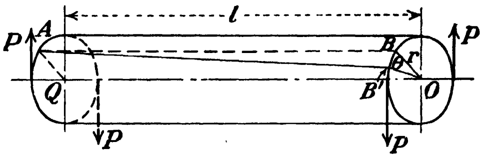


FIG. 53.

Also, since

$$T = \frac{S_s J}{r},$$

$$T = \frac{E_s J\theta}{l}.$$

PROBLEMS

1. A brass spindle ½ in. in diameter and 3 in. long is subjected to a torque of 84 in.-lb. Through what angle is one end twisted with respect to the other? (See Table VIII for values of E_s.) *Ans.* 0° 29′ 25″.

2. If the shaft described in Problem 2, Art. 32, is 40 ft. long, how much is one end twisted with respect to the other? *Ans.* 2° 52′.

3. A steel shaft 10 ft. long is to transmit a torque of 48,000 in.-lb. The allowable shearing stress is 8000 lb./in.^{2} and the twist must not exceed 1°. Compute the diameter necessary. *Ans.* 4.09 in.

4. A shaft 6 ft. long is 2 in. outside diameter and 1.75 in. inside diameter. How much is one end twisted with respect to the other when stressed to 8000 lb./in.^{2} shearing stress? *Ans.* 2° 45′.

5. If the shearing elastic limit of aluminum wire is 12,000 lb./in.^{2}, in what length may a wire 0.1 in. in diameter be twisted through one complete revolution without injury? *Ans.* 8.73 ft.

34. Transmission of Power by a Shaft.—It is shown in mechanics that the work of a rotating force upon a rotating body is $T\theta$, T being the torque of the force and θ the angle through which the rotating body turns. If a shaft is being rotated at a constant speed against a resistance, the work done in one revolution is $2\pi T$. If T is in units of inch-pounds and if N is the number of revolutions per minute, the work done in a minute is $2\pi TN$ inch-pounds. Since 396,000 inch-pounds of work per minute constitute one horsepower, the number of horsepower, represented by hp, is given by the expression,

$$hp = \frac{2\pi TN}{396{,}000}.$$

$$T = 63{,}025\,\frac{hp}{N}.$$

PROBLEMS

1. If allowable $S_s = 9000$ lb./in.^{2}, how many horsepower will a 6-in. shaft transmit when rotating at 180 r.p.m.? *Ans.* 1090 hp.

2. Compute the diameter of a shaft to transmit 6000 hp. at 180 r.p.m., with allowable $S_s = 8000$ lb./in.^{2}. *Ans.* 11 in.

3. If the inside diameter of a hollow shaft is to be ¾ of the outside diameter, what must be the size of shaft to transmit 120 hp. at 3600 r.p.m. with an allowable stress of 7500 lb./in.^{2}? *Ans.* 1.278 in.

4. A steel shaft 2 in. in diameter is transmitting 358 hp. at 2400 r.p.m. Through what angle is it twisted in a length of 5 ft.? *Ans.* 1°43′.

35. Shearing Stresses in Noncircular Shafts.—When a circular bar is twisted, each cross section of the bar remains a plane circle, as may be shown by tests. When a noncircular bar is twisted, however, any original cross section of the bar does not remain plane, but is warped, as can be seen by ruling rectangles on bars of rubber of rectangular or ellipsoidal cross section and twisting them. On account of the warping of the cross sections, the rectangles at the middle of the long side of the rectangular bar have the largest shearing distortion and therefore the material at this point has the largest shearing stress, while that at the corners approaches zero. At the middle of the short side of the cross section there is some distortion in shear, but of smaller amount than that at the middle of the longer side.

Similarly, the rectangular figures at the ends of the shorter axis of the bar with an elliptical cross section have a larger shearing distortion than those at the ends of the longer axis, indicating a larger shearing stress there.

Saint Venant made a complete analysis of the distribution of the shearing stresses in a bar with a rectangular cross section, and for a section with long side a and short side b, developed the following expression for the maximum shearing stress, T being the torque.

$$\text{Max. } S_s = \frac{3a + 1.8b}{a^2b^2}T.$$

For a square section, $b = a$ and

$$\text{max. } S_s = \frac{4.8T}{a^3}.$$

For a circular section of diameter a,

$$S_s = \frac{5.1T}{a^3}.$$

It will be seen from this that with the same torque, the maximum stress in a bar with a square section is only 6 per cent smaller than that in a bar with a circular section of the size of the inscribed circle, while the weight is 27 per cent greater.

PROBLEMS

1. Compute the size of a square shaft to transmit a torque of 10,000 in.-lb. with an allowable shearing stress of 5000 lb./in.^{2}. *Ans.* 2.125 in.

2. Compute the diameter of a circular shaft to transmit the same torque with the same stress as that given in Problem 1. *Ans.* 2.17 in.

3. Design a rectangular shaft, with side $a = 4b$, to transmit the same torque with the same stress as that given in Problem 1 and compare its weight with that of the circular shaft. *Ans.* 4.8 by 1.2 in.; 1.56 to 1.

4. Compare the torsional resistance of a bar 6 by ¼ in. in cross section with that of a hollow circular bar made by forming the plate into a tube, butt-welded, with a mean circumference of 6 in. *Ans.* 1 to 10.58.

CHAPTER V

SHEAR AND MOMENT IN BEAMS

36. Loads and Supports on Beams.—A beam may be defined as a bar of material upon which are acting transverse loads and reactions. In buildings, bridges, and such structures, beams are usually horizontal and the loads are usually weights. In machines, beams may have any direction and the loads are often pressures from other parts of the machine or from other outside bodies.

Loads are classified as (1) concentrated loads; (2) uniformly distributed loads; and (3) varyingly distributed loads. Wheel loads and the pressures of floor beams on girders are examples of concentrated loads. Reactions on beams are usually considered to be concentrated, the point of application being the center of the area of contact. Distributed loads may be of uniform intensity or of varying intensity. Uniformly distributed loads are much more common than loads of varying intensity. Water pressure on a vertical wall and the weight of piles of granular material such as sand, grain, or cement piled high at one side of a bin and sloping down to zero at the other side, are examples of loads of varying intensity.

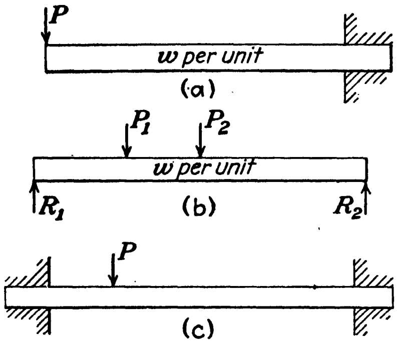


FIG. 54.

A beam which is fixed at one end and free at the other is called a *cantilever* beam, as shown in Fig. 54(*a*). A beam which is supported at the ends is called a *simple* beam, as shown in Fig. 54(*b*). The beam shown in Fig. 54(*c*) is *fixed* at the ends. A beam may be continuous over several supports and it may overhang a support, as shown in Fig. 55.

In the case of simple beams and of overhanging beams with two supports, it is necessary to compute the reactions by the principle of moments.

Since the beam which is acted upon by its loads and reactions is in equilibrium, the summation of moments of all of the external forces with respect to any axis equals zero. If this axis of moments is chosen on the line of action of one of the reactions, the equation of moments will determine the reaction at the other support. A second moment equation will then determine the remaining reaction. The equation $\Sigma F_y = 0$ will give a check solution. If the loading is symmetrical, each reaction is one-half of the total load.

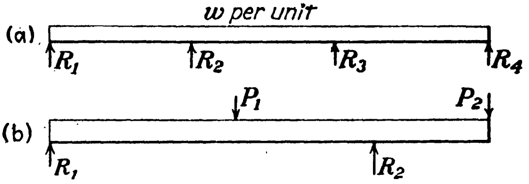


FIG. 55.

PROBLEMS

1. A timber beam 4 in. wide, 16 in. deep, and 30 ft. long is supported at the ends and carries a load of 250 lb. per linear ft. in addition to its own weight. Compute the reactions. (See Table I for the weight of timber.) *Ans.* 3990 lb.

2. A 12-in. 31.8-lb. structural I-beam 20 ft. long carries a superimposed uniformly distributed load of 400 lb. per linear ft. and a concentrated load of 6800 lb. at the middle. Compute the reactions. *Ans.* 7718 lb.

3. Solve Problem 2 if the left support is moved to a point 4 ft. from the left end. *Ans.* 9648 lb.; 5788 lb.

4. A simple reinforced concrete beam is 6 ft. wide, 4 ft. deep, and 30 ft. long. In addition to its own weight it carries a uniformly distributed load of 250 lb./ft.^{2} of floor area, a concentrated load of 20,000 lb. 6 ft. from the left end, and another of 44,000 lb. at the middle. Compute the reactions. *Ans.* 114,500 lb.; 102,500 lb.

5. A beam 40 ft. long weighing 112 lb. per linear ft. is supported at the left end and at a point 6 ft. from the right end. It carries a brick masonry wall 16 in. thick which is 12 ft. high for a distance of 10 ft. at the left end, and then slopes down to 6 ft. high at the right end. Compute the reactions. *Ans.* 31,780 lb.; 35,100 lb.

37. Shearing Forces in Beams.—The transverse loads and reactions acting upon a beam have a tendency to shear it off at any cross section. In Fig. 56, consider as a free body the part of the beam on the left of the section *MN*. The free body with the forces acting upon it is shown in Fig. 56(*b*). The forces external to the beam are *P* and wl_1, both acting downward. In order that this free body may be held in equilibrium, there must be added to it at the section *MN* the internal forces as shown. These must consist of a tensile force *T* in the upper part of the section, a compressive force *C* in the lower part of the section, and a vertical shearing force *V* acting upward in the section. Since the forces *T* and *C* have no vertical components, and since the vertical summation of the forces on the free body must equal zero,

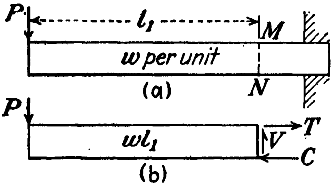


FIG. 56.

$$V - P - wl_1 = 0.$$
$$V = P + wl_1.$$

Consider next a simple beam, Fig. 57, weighing 1000 pounds and supporting a load of 2000 pounds at the middle. In Fig. 57(*b*) is shown as a free body the part of the beam on the left of section *MN*. The upward reaction at the left end is 1500 pounds and the downward force of the weight of this part is 200 pounds. It is seen that it is necessary for equilibrium that the shearing force V_2 at the section *MN* shall act downward. As before, the summation of the vertical forces must equal zero, so

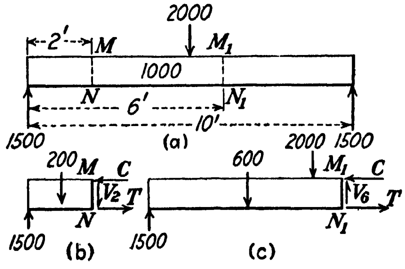


FIG. 57.

$$V_2 + 200 - 1500 = 0.$$
$$V_2 = 1300 \text{ pounds}.$$

If the part of the beam on the left of section M_1N_1 is considered as the free body, as shown in Fig. 57(*c*), the shearing force V_6 is upward.

$$V_6 + 1500 - 2600 = 0.$$
$$V_6 = 1100 \text{ pounds.}$$

When the expression "vertical shear at a section" is used, the internal shear is meant, but its amount is determined by the vertical summation of the external forces on the beam on the left of that section. The sign of the vertical shear at any section is determined from the direction of the vertical summation of the forces on the left of that section. The shear is said to be positive if this vertical summation of forces is upward, and negative if it is downward. The shear at section MN is therefore positive; that at section M_1N_1 is negative.

PROBLEMS

1. In Fig. 56, $P = 800$ lb., $w = 320$ lb. per linear ft., and $l_1 = 12$ ft. Compute the vertical shear at the section MN. *Ans.* -4640 lb.

2. In the beam shown in Fig. 57, compute the vertical shear at the left of the load, at the right of the load, and at the left of the right reaction.

Ans. $+1000$ lb.; -1000 lb.; -1500 lb.

3. A simple beam 6 ft. long weighs 30 lb. Compute the shear just inside the left reaction and at each foot along the beam.

Ans. $V_0 = 15$ lb.; $V_1 = 10$ lb.; $V_2 = 5$ lb.; $V_3 = 0$; $V_4 = -5$ lb.; $V_5 = -10$ lb.; $V_6 = -15$ lb.

38. Shear Equations and Shear Diagrams.—It was noticed in the solution of the problems in Art. 37 that there is a regular variation in the amount of the shear from left to right across the beam until a concentrated load or reaction is reached. With downward loads, this variation is always a reduction, the amount of the reduction per unit length being the weight of the uniformly distributed load per unit of length. At a concentrated load or at a reaction, there is a sudden change in the amount of the shear, the change being the same in amount and direction as the load or reaction.

The variation in the shear may be shown either algebraically by an equation, or graphically by a diagram. The beam shown in Fig. 58 is a cantilever beam fixed at the right end and loaded with a uniformly distributed load of w per unit of length. The vertical shear at any section at a distance x from the left end is

$-wx$, so the equation $V = -wx$ gives the relation between the shear and the distance x from the left end of the beam.

The shear diagram, Fig. 58(b), shows graphically the same variation. The line OX is the shear axis, drawn to represent to scale the length of the beam. From this axis, vertical ordinates are laid off to represent to some scale the amount of the vertical shear at that section in the beam. If a line OA is drawn through the ends of these ordinates, and V is used as the variable vertical ordinate, the equation of the line becomes $V = -wx$.

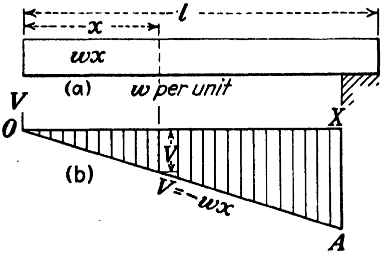


FIG. 58.

In Fig. 59, a cantilever beam with both uniformly distributed and concentrated loads is shown. The equation of the shear is $V = -500 - 30x$. In this case the uniformly distributed load is 30 pounds per linear foot, and the distance x must be in feet. If with the same load, the amount of the uniform load is used as 2.5 pounds per linear inch, the distance x must be in inches. In the shear diagram, the ordinate at the left represents

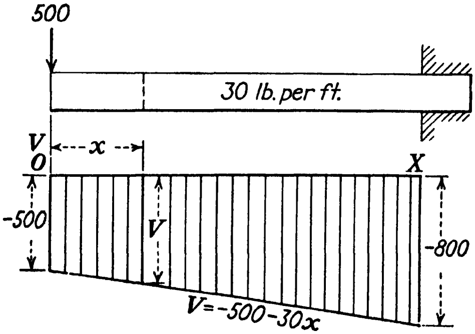


FIG. 59.

500 pounds to scale, the ordinate at the middle represents 650 pounds to scale, and that at the fixed end represents 800 pounds to scale.

The diagram shown in Fig. 60 is one for a simple beam of length l loaded with a weight of w per unit length. The reactions

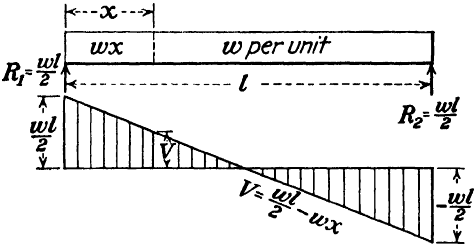


FIG. 60.

are each $wl/2$ and the equation of the line joining the ends of the shear ordinates is $V = (wl/2) - wx$.

If a beam carries one or more concentrated loads and its own weight is extremely small, or for any other reason is neglected, the shear curve is a straight line parallel to the axis. Such diagrams are shown in Figs. 61 and 62.

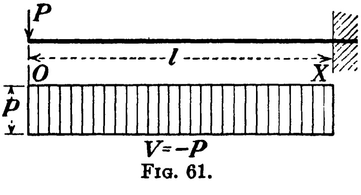


FIG. 61.

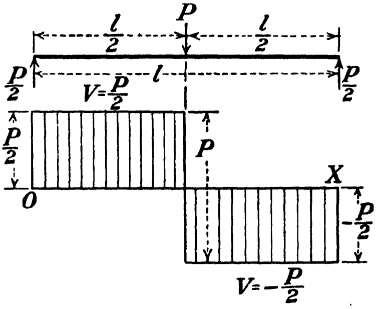


FIG. 62.

The values of the shear at the controlling points should be shown at the proper places on the shear diagram. If the shear diagram crosses the axis at a point other than a reaction or a concentrated load, the distance from the end of the beam should also be shown. In Fig. 63 is shown the shear diagram for a beam 24 feet long, weighing 20 pounds per linear foot, and carrying in addition a concentrated load of 800 pounds at a point 4 feet from the left end and another of 200 pounds at a point 2 feet from the right end. The beam is supported at the left end and at a point 4 feet from the right end. The left reaction is 812 pounds and the right reaction is 668 pounds.

For this diagram there are four equations for the four curves joining the ends of the shear ordinates, as follows:

From $x = 0$ to $x = 4$, $V = 812 - 20x$;
From $x = 4$ to $x = 20$, $V = 12 - 20x$;
From $x = 20$ to $x = 22$, $V = 680 - 20x$;
From $x = 22$ to $x = 24$, $V = 480 - 20x$.

It is seen from all of the preceding discussion that the maximum numerical value of the shear in a cantilever beam is always at the fixed end. In a simple beam or in an overhanging beam with two supports, the maximum numerical value of the shear occurs at a point next to one or the other of the supports.

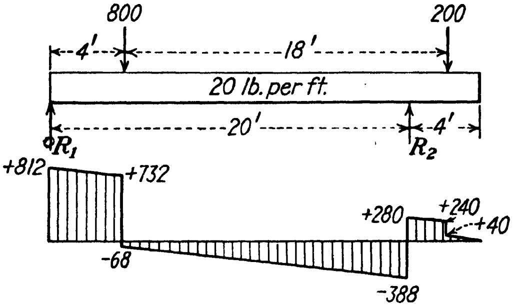


FIG. 63.

PROBLEMS

1. A cantilever beam 12 ft. long is fixed at the right end and carries a load of 6000 lb. at the left end. Neglecting the weight of the beam, draw the shear diagram to the scale of 1 in. = 2 ft. and 1 in. = 3000 lb. Write the shear equation. *Ans.* $V = -6000$ lb.

2. A cantilever beam 10 ft. long weighing 142 lb. per linear ft. is fixed at the right end. Draw the shear diagram to the scale of 1 in. = 2 ft. and 1 in. = 1000 lb. Write the shear equation. *Ans.* $V = -142x$.

3. A cantilever beam 16 ft. long is fixed at the right end and carries a uniformly distributed load of 60 lb per linear ft., and a concentrated load of 3000 lb. 6 ft. from the left end Draw the shear diagram and write the two shear equations.

Ans. 0 to 6 ft., $V = -60x$; 6 to 16 ft., $V = -3000 - 60x$.

4. A simple beam 20 ft. long carries a concentrated load of 4600 lb. at the middle. Draw the shear diagram to the scale of 1 in. = 4 ft. and 1 in. = 2000 lb. Write the shear equations.

Ans. 0 to 10 ft., $V = +2300$ lb.; 10 to 20 ft., $V = -2300$ lb.

5. A simple beam 80 ft. long carries a uniformly distributed load of 4800 lb. per linear ft. Draw the shear diagram to the scale of 1 in. = 20 ft. and 1 in. = 200,000 lb. Write the shear equation.

Ans. $V = 192{,}000 - 4800x$.

6. Solve Problem 5 if it carries in addition a concentrated load of 120,000 lb. at a point 25 ft. from the left end.

Ans. 0 to 25 ft., $V = 274{,}500 - 4800x$; 25 to 80 ft., $V = 154{,}500 - 4800x$.

7. A beam 24 ft. long is supported at the left end and at a point 4 ft. from the right end. It carries a uniformly distributed load of 400 lb. per linear ft., a concentrated load of 7200 lb. 6 ft. from the left end, and one of 2000 lb. at the right end. Draw the shear diagram to the scale of 1 in. = 4 ft. and 1 in. = 6000 lb.

8. A simple beam 30 ft. long weighs 100 lb. per linear ft. and carries a superimposed load which varies uniformly from zero at the left end to a value of 1200 lb. per linear ft. at the right end. Write the shear equation and draw the shear diagram. *Ans.* $V = 7500 - 100x - 20x^2$.

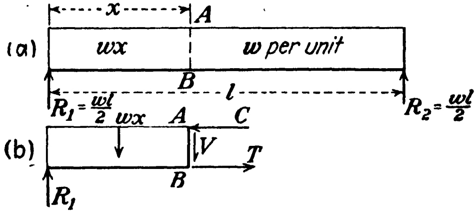


FIG. 64.

39. Moment in Beams.—In addition to the shearing action exerted upon a beam by its transverse loads and reactions, there is also a bending action. In Fig. 64(*a*) the simple beam is supported at the ends by the upward reactions R_1 and R_2, and is pressed downward by the uniformly distributed load wl. If the part of the beam on the left side of the section AB is considered as the free body, as shown in Fig. 64(*b*), there are acting upon it the external forces R_1 upward at the left end and the weight wx downward at the middle. These forces have a tendency to rotate the free body about a horizontal axis in the cross section, with a moment the amount of which is $wlx/2 - wx^2/2$. This moment is called the *moment of the external forces,* or the *external moment* at the section AB. In order to hold the free body in equilibrium it is necessary to add at the section AB the shearing force V already discussed and the pair of compressive and tensile forces C and T, C acting in the upper part of the section and T acting in the lower part of the section. Since the forces C and

T are the only horizontal forces acting upon the free body, and since the free body is in equilibrium, the summation of horizontal forces must equal zero, so $C - T = 0$, or $C = T$. The forces C and T, therefore, form a couple, and the moment of this couple is called the *internal moment* in the beam at the section AB. The internal moment is necessarily equal in amount to the external moment and opposite in direction, since the two moments must balance each other on the free body.

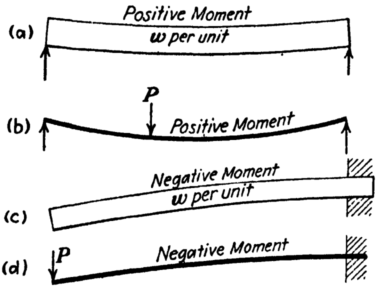


FIG. 65.

When the expression "moment at a section" is used, the internal moment is meant, but its amount must always be determined from the amount of the external moment.

The moment at any section is said to be positive when the moment of the external forces on the left of the section is clockwise, and negative when it is counterclockwise. If the loading of a beam is such that the beam is concave upward, the moment is positive, as shown in Figs. 65(a) and (b). If the beam is convex upward, the moment is negative, as shown in Figs. 65(c) and (d).

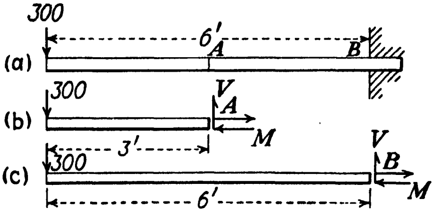


FIG. 66.

If in Fig. 64 the portion of the beam on the right of the section AB is considered as the free body, the external moment of the forces on the right of the section is $\frac{wl}{2}(l - x) - \frac{w}{2}(l - x)^2$. This expression, if simplified, becomes $wlx/2 - wx^2/2$ as above.

EXAMPLE 1

A cantilever beam 6 ft. long has a load of 300 lb. at the free end. Compute the moment at the middle and at the fixed end due to this load.

Solution.—Figure 66(*a*) shows the beam loaded as described. Figure 66(*b*) shows the part of the beam on the left of the middle as a free body. The moment of the external forces on the left of the section at A, with respect to any horizontal axis in the section, is $300 \times 3 = 900$ ft.-lb. Since the beam is bent convex upward, the moment is negative, so $M_A = -900$ ft.-lb. or $-10{,}800$ in.-lb.

In Fig. 66(*c*) is shown the entire beam as a free body. The moment of the external forces on the left of the section at B with respect to any horizontal axis in the section is $300 \times 6 = 1800$ ft.-lb., so $M_B = -1800$ ft.-lb. or $-21{,}600$ in.-lb.

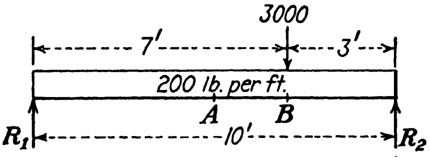


FIG. 67.

EXAMPLE 2

A beam 10 ft. long weighing 200 lb. per linear ft. is supported at the ends and carries a concentrated load of 3000 lb. at a point 7 ft. from the left end, as shown in Fig. 67. Compute the moment at the middle of the beam and under the load.

Solution.—Consider first the entire beam as a free body. The moment equation with respect to an axis at the right reaction gives

$$10R_1 = 2000 \times 5 + 3000 \times 3.$$
$$R_1 = 1900 \text{ lb.}$$
$$R_2 = 5000 - 1900 = 3100 \text{ lb.}$$

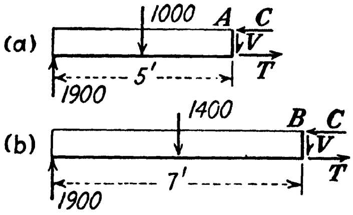


FIG. 68.

Consider the left half of the beam as the free body, as shown in Fig. 68(*a*). The moment with respect to point A of all of the external forces on the beam on the left of the section is

$$M_A = 1900 \times 5 - 1000 \times 2.5.$$
$$M_A = 7000 \text{ ft.-lb., or } 84{,}000 \text{ in.-lb.}$$

Consider the part of the beam on the left of the concentrated load as the free body, as shown in Fig. 68(*b*). The moment with respect to point B of all of the external forces on the beam on the left of the section is

$$M_B = 1900 \times 7 - 1400 \times 3.5.$$
$$M_B = 8400 \text{ ft.-lb. or } 100{,}800 \text{ in.-lb.}$$

EXAMPLE 3

Compute the moment at 5, at 10, at 15, and at 20 ft. from the left end of the beam shown in Fig. 69.

Solution.—The moment equation with respect to an axis at R_2 for the entire beam as the free body gives

$$20R_1 = 1000 \times 15 + 2400 \times 8 - 1000 \times 4.$$
$$R_1 = 1510 \text{ lb.}$$

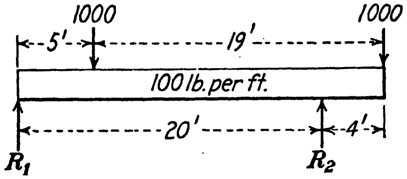


FIG. 69.

With the 5-ft. length at the left end of the beam as the free body, as shown in Fig. 70(*a*).

$$M_5 = 1510 \times 5 - 500 \times 2.5 = 6300 \text{ ft.-lb.}$$

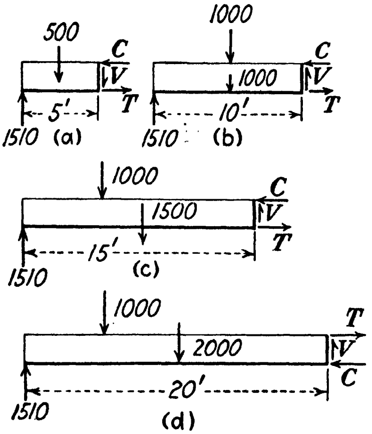


FIG. 70.

With the 10-ft. length at the left end of the beam as the free body, as shown in Fig. 70(*b*),

$$M_{10} = 1510 \times 10 - 1000 \times 5 - 1000 \times 5 = 5100 \text{ ft.-lb.}$$

With the 15-ft. length at the left end of the beam as the free body, as shown in Fig. 70(*c*),

$$M_{15} = 1510 \times 15 - 1000 \times 10 - 1500 \times 7.5 = 1400 \text{ ft.-lb.}$$

With the 20-ft. length at the left end of the beam as the free body, as shown in Fig. 70(*d*),

$$M_{20} = 1510 \times 20 - 1000 \times 15 - 2000 \times 10 = -4800 \text{ ft.-lb.}$$

The moment at the 20-ft. point is negative so the beam at this point is convex upward. Force *C* at the section is acting at the bottom and force *T* at the top.

PROBLEMS

1. A cantilever beam 15 ft. long is fixed at the right end and carries a load of 2000 lb. at the free end and one of 800 lb. 4 ft. from the free end. Compute the moment at the fixed end due to these loads.

Ans. −38,800 ft.-lb.

2. In Example 3 above, check the value of M_{20} by using the 4-ft. length at the right end of the beam as the free body.

3. Compute the moment at a section 2 ft. from the left end and also at the fixed end in the beam shown in Fig. 71.

Ans. $M_2 = -2640$ ft.-lb.; $M_{12} = -24,540$ ft.-lb.

4. A simple beam 20 ft. long weighing 180 lb. per linear ft. carries a load of 40,000 lb. at the middle. Compute the moment at a point 5 ft. from the left end and at the middle.

Ans. $M_5 = 106,750$ ft.-lb.; $M_{10} = 209,000$ ft.-lb.

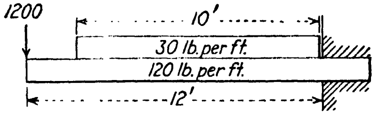


FIG. 71.

5. For the beam shown in Fig. 72, compute the moment at 4, at 8, at 12, and at 12.5 ft. from the left end.

Ans. $M_4 = -3200$ ft.-lb.; $M_8 = 7200$ ft.-lb.; $M_{12} = 11,200$ ft.-lb.; $M_{12.5} = 11,250$ ft.-lb.

40. Moment Equations and Moment Diagrams.

—It was shown in Art. 38 that the variation in the shear in a beam could be shown either algebraically by an equation or graphically by a diagram. The variation in the moment may be shown in a similar manner.

With a part of the beam x units long at the left end of the beam as the free body, the expression is written for the moment about the right end of the free body of all of the external forces acting upon it. At concentrated loads and at reactions, the expression for the moment changes.

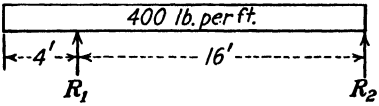


FIG. 72.

In order to show the variation in the moment by a diagram the length of the beam is laid off along the X axis according to some scale. From the X axis, ordinates are laid off at various points representing to scale the moments at the corresponding sections in the beam. A smooth curve drawn through the ends of these ordinates completes the diagram.

EXAMPLE 1

A cantilever beam 10 ft. long is fixed at the right end and carries a load of 800 lb. at the left end. Write the moment equation and draw the moment diagram.

Solution.—At any point distant x from the left end of the beam, the moment of all of the external forces on the left of the section with respect to a horizontal axis in the section is $800x$. The beam is convex upward, so the moment is negative. The moment equation is therefore

$$M = -800x.$$

This is the equation of a straight line passing through the origin and sloping downward to the right.

In Fig. 73(*a*) the beam is drawn to a scale of 1 in. = 5 ft. Immediately below it, in Fig. 73(*b*), the line *OA* is laid off to the same scale. Since the moment curve is a straight line and passes through the origin, one other point is sufficient to locate it completely. At the fixed end the moment is −8000 ft.-lb. This value is laid off to the scale of 1 in. = 10,000 ft.-lb., in the downward direction since it is negative, and locates point *B*. The line *OB* is then the moment curve, and the moment at any section is given by the length of the ordinate between the *X* axis and the line *OB*.

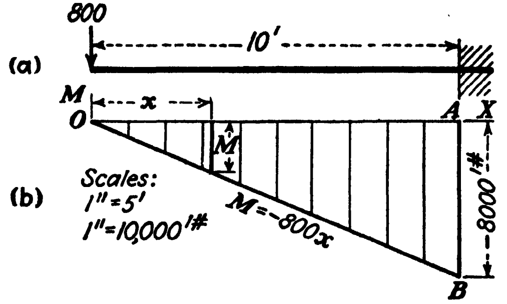


FIG. 73.

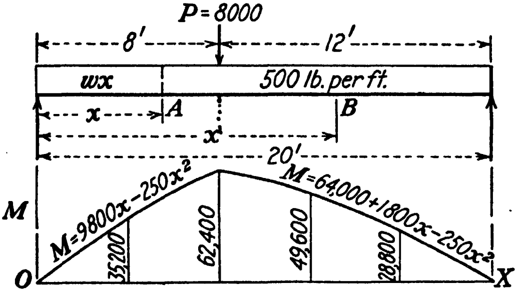


FIG. 74.

EXAMPLE 2

A simple beam 20 ft. long carries a uniformly distributed load of 500 lb. per linear ft. and a concentrated load of 8000 lb. at a point 8 ft. from the left end as shown in Fig. 74. Write the moment equations and draw the moment diagram.

Solution.—The reaction R_1 is given by the equation

$$20R_1 = 10{,}000 \times 10 + 8000 \times 12.$$
$$R_1 = 9800 \text{ lb.}$$
$$R_2 = 18{,}000 - 9800 = 8200 \text{ lb.}$$

For any value of x between 0 and 8 ft., the moment about point A of all of the forces on the beam on the left of point A is,

✓
$$M_A = 9800x - 250x^2.$$

For any value of x between 8 and 20 ft., the moment about point B of all of the forces on the beam on the left of point B is,

$$M_B = 9800x - 8000(x - 8) - 250x^2.$$

This equation reduces to

$$M_B = 64{,}000 + 1800x - 250x^2.$$

In this case the moment curves are not straight lines, so the moment at each 4-ft. point will be computed in order to draw the moment diagram. The moment at each end is zero.

$$\begin{aligned}
M_4 &= 9800 \times 4 - 250 \times 16 = 35{,}200 \text{ ft.-lb.}\\
M_8 &= 9800 \times 8 - 250 \times 64 = 62{,}400 \text{ ft.-lb.}\\
M_{12} &= 64{,}000 + 1800 \times 12 - 250 \times 144 = 49{,}600 \text{ ft.-lb.}\\
M_{16} &= 64{,}000 + 1800 \times 16 - 250 \times 256 = 28{,}800 \text{ ft.-lb.}
\end{aligned}$$

In Fig. 74(*b*) ordinates at the 4-ft. points are laid off from the axis OX to represent to scale these values of the moments. The diagram is completed by drawing through the upper ends of these ordinates the two smooth curves. The moment at any section in the beam is given by the scaled value of the corresponding ordinate.

PROBLEMS

1. A cantilever beam 8 ft. long, fixed at the right end, carries a uniformly distributed load of 1000 lb. per linear ft. Draw the beam and the moment diagram, using scales of 1 in. = 2 ft and 1 in. = 20,000 ft.-lb. moment.

2. Using the same scale, draw the moment diagram for the beam described in Problem 1 if a concentrated load of 2000 lb. is added at the middle of the beam.

3. A simple beam 8 ft. long carries a load of 120 lb. at the middle. Write the two moment equations and draw the moment diagram.

Ans. $M_A = 60x$; $M_B = 480 - 60x$.

4. A simple beam 40 ft. long carries a uniformly distributed load of 1800 lb. per linear ft. Write the moment equation and draw the moment diagram to the scale of 1 in. = 8 ft. and 1 in. = 200,000 ft.-lb.

Ans. $M = 36{,}000x - 900x^2$.

5. A simple beam 20 ft. long carries a load which varies uniformly from zero at the left end to a value of 2400 lb. per linear ft. at the right end. Write the moment equation. *Ans.* $M = 8000x - 20x^3$.

41. Relation between Shear and Moment.—Let Fig. 75 represent a beam with any kind of loading. Let x be the distance from the left end to any section A, and let dx be the differential length at point A. In Fig. 76 this differential length is shown as a free body, drawn to a larger scale. The forces acting upon the free body consist of the following: its own weight, wdx; the vertical shears V and V_1 and the moments M and M_1 at the two sections. Either the moments or the shearing forces, or both, may be reversed in direction. Since this free body is in equilibrium, the moment of all of the forces on the body with respect to any axis is equal to zero. The moment equation with the point C as the axis gives

$$M_1 - M - Vdx + \frac{w(dx)^2}{2} = 0.$$

Since the quantity $M_1 - M$ is the increment in the moment in the differential length dx, it is the differential moment dM. The fourth term, $w(dx)^2/2$, is a differential of a higher order and is therefore negligible. The expression then becomes,

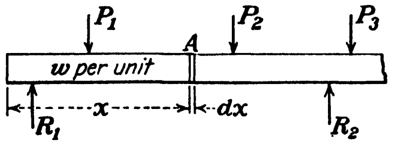


FIG. 75.

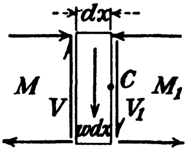


FIG. 76.

$$dM = Vdx.$$

This relation may also be written,

$$\frac{dM}{dx} = V.$$

If this expression in the form $dM = Vdx$ is integrated, it becomes,

$$\int_{M_1}^{M_2} dM = \int Vdx.$$

$$M_2 - M_1 = \int Vdx.$$

It will be seen in the shear diagram, Fig. 77(*b*), that the quantity Vdx is the differential area of the shear diagram. The integral of the quantity Vdx between any two values of x will give the area of the shear diagram between these two values of x.

The area of the shear diagram between any two values of x is equal to the change in moment in the beam in the same length.

Since the moment at the end of a beam is zero, *the total area of the shear diagram up to any point is equal to the moment at that point.*

The area of the shear diagram above the axis is positive and the area below the axis is negative. Since the sum of all of the area of the shear diagram from the left end to the right end must equal the moment at the right end which is zero, it follows that there must be as much positive area as negative area. Likewise, the area of the shear diagram on the right of any point must be equal in amount to the area on the left of the same point, but must be opposite in sign. It follows that if the area of the shear diagram on the right of any section is used to obtain the moment at that section, the sign must be changed. Area above the axis on the right of any section indicates negative moment at that section, and area below the axis indicates positive moment.

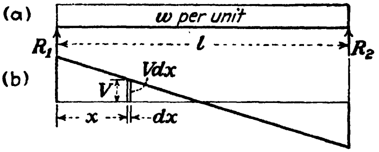


FIG. 77.

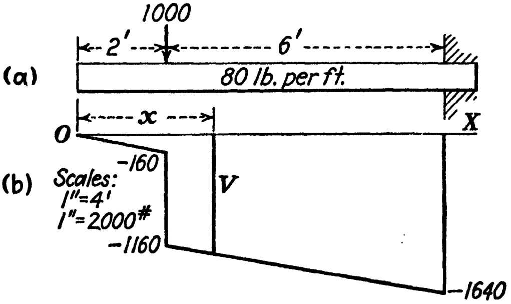


FIG. 78.

Since the shear diagram shows both shear and moment, the shear by the ordinates and the moment by the area, it is usually the only diagram drawn.

EXAMPLE 1

A cantilever beam 8 ft. long carries a uniformly distributed load of 80 lb. per linear ft. and a concentrated load of 1000 lb. at a point 2 ft. from the free end. Draw the shear diagram and compute the moment at the fixed end.

Solution.—Figure 78(*a*) shows the beam drawn to a scale of 1 in. = 4 ft. Figure 78(*b*) shows the shear diagram drawn to a scale of 1 in. = 2000 lb.

The moment at the fixed end is given by the area of the shear diagram, all of which is negative. The area of the triangle is $-\frac{1}{2} \times 2 \times 160 = -160$ ft.-lb. The area of the trapezoid is $-\frac{1160 + 1640}{2} \times 6 = -8400$ ft.-lb. The total area, and therefore the moment at the fixed end, is $-160 - 8400 = -8560$ ft.-lb.

It should be noted that this shear diagram does not extend to the right end of the beam where the moment is zero.

EXAMPLE 2

Draw the shear diagram for the beam shown in Fig. 79(*a*) and compute the moment at each 5-ft. point.

Solution:

$$\begin{aligned} 15R_1 &= 2400 \times 10 + 3000 \times 17 + 6000 \times 6. \\ R_1 &= 7400 \text{ lb.} \\ R_2 &= 11{,}400 - 7400 = 4000 \text{ lb.} \end{aligned}$$

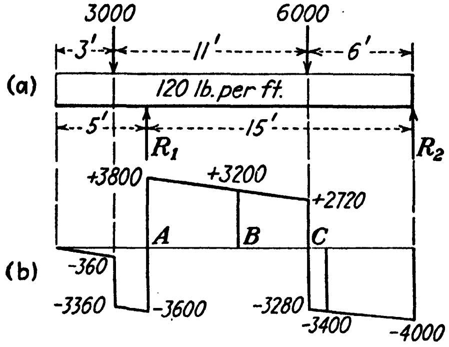


FIG. 79.

The shears at the controlling points are shown in the diagram, Fig. 79(*b*). The moment at point *A*, 5 ft. from the left end, is given by the area of the triangle and the trapezoid on the left of point *A*.

$$M_A = -\tfrac{1}{2} \times 360 \times 3 - \frac{3360 + 3600}{2} \times 2 = -7500 \text{ ft.-lb.}$$

$$M_B = -7500 + \frac{3800 + 3200}{2} \times 5 = +10{,}000 \text{ ft.-lb.}$$

$$M_C = -7500 + \frac{3800 + 2720}{2} \times 9 - \frac{3280 + 3400}{2} \times 1 = +18{,}500 \text{ ft.-lb.}$$

The value of M_C may be obtained more easily by getting the area of the shear diagram on the right of point *C*. The area is negative, so the moment is positive.

$$M_C = \frac{3400 + 4000}{2} \times 5 = +18{,}500 \text{ ft.-lb.}$$

PROBLEMS

1. A cantilever beam 10 ft. long is fixed at the right end and carries a uniformly distributed load of 2400 lb. total. Draw the shear diagram and compute the moment at the middle and at the fixed end.

Ans. −3000 ft.-lb.; −12,000 ft.-lb.

2. A cantilever beam 12 ft. long is fixed at the right end and carries a uniformly distributed load of 300 lb. per linear ft. and a concentrated load of 1200 lb. at the free end. Draw the shear diagram and compute the moment at the fixed end.

Ans. −36,000 ft.-lb.

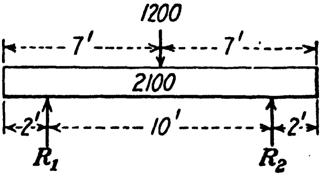


FIG. 80.

3. An 18-in. 60-lb. I-beam 15 ft. long is used as a cantilever beam. In addition to its own weight it carries a load which varies uniformly from zero at the free end to a value of 1500 lb. per linear ft. at the fixed end. Draw the shear diagram and compute the moment at the fixed end.

Ans. −63,000 ft.-lb.

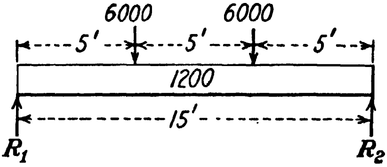


FIG. 81.

4. A simple beam 20 ft. long carries a uniformly distributed load of 1200 lb. per linear ft. and a concentrated load of 5000 lb. at the middle. Draw the shear diagram and compute the moment under the load.

Ans. 85,000 ft.-lb.

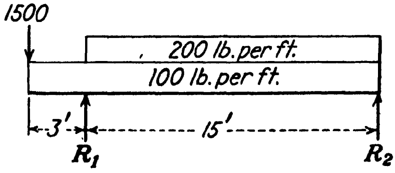


FIG. 82.

5. A beam is supported and loaded as shown in Fig. 80. Draw the shear diagram and compute the moment at the middle. *Ans.* 4575 ft.-lb.

6. A beam is supported and loaded as shown in Fig. 81. Draw the shear diagram and compute the moment under the left load and at the middle.

Ans. 32,000 ft.-lb.; 32,250 ft.-lb.

7. A beam is supported and loaded as shown in Fig. 82. Draw the shear diagram and compute the moment at the two points where the shear diagram crosses the axis. *Ans.* $M_3 = -4950$ ft.-lb.; $M_{11.6} = 6144$ ft.-lb.

8. A simple beam 30 ft. long weighing 320 lb. per linear ft. carries a superimposed load which varies uniformly from zero at the left end to a value of 2000 lb. per linear ft. at a point 20 ft. from the left end, and then continues at 2000 lb. per linear ft. to the right end. Draw the shear diagram and compute the moment at the point where the shear diagram crosses the axis. *Ans.* $M_{16.7} = 199{,}800$ ft.-lb.

42. Sections of Maximum Moment.

42. Sections of Maximum Moment.—In the case of moment as in the case of shear, the largest numerical value regardless of sign is called the maximum. The positive sign merely indicates that the beam is concave upward, and the negative sign that it is convex upward. It has been noticed in the solution of the problems in Art. 41 that the value of the moment begins at zero at the left end of the beam and increases either positively or negatively according as the area of the shear diagram is positive or negative until a section is reached at which the shear passes through zero and therefore changes sign. At this section the moment has reached a numerical maximum value and then begins to decrease, since from this point to the right, shear area of the opposite sign is added on. The value of the moment may pass through zero and reach a second numerical maximum value of opposite sign if the shear diagram again passes through zero.

Sections of maximum moment are sometimes called *danger sections* or *dangerous sections.*

The four cases tabulated below occur so often in engineering practice that their values should be memorized.

Beam	Load	Maximum moment	
		Position	Amount
Cantilever..................	P at end	Fixed end	Pl
Cantilever..................	W distributed	Fixed end	$\frac{Wl}{2}$
Simple.....................	P at middle	Middle	$\frac{Pl}{4}$
Simple.....................	W distributed	Middle	$\frac{Wl}{8}$

EXAMPLE

Compute the reactions of the beam shown in Fig. 83(*a*). Draw the shear diagram and locate the points of maximum moment. Compute the maximum moment and locate the point where the moment is zero. Draw the moment diagram.

Solution:

$$10R_1 = 2800 \times 7.$$
$$R_1 = 1960 \text{ lb.}$$
$$R_2 = 2800 - 1960 = 840 \text{ lb.}$$

The shear diagram is shown in Fig. 83(*b*). At point *A* the shear is zero. From *A* to *B* the shear increases negatively and at *B* becomes $-4 \times 200 = -800$ lb. At this point the shear increases positively by the amount of $R_1 = 1960$ lb. and becomes $-800 + 1960 = +1160$ lb. just at the right of the reaction. From *B* to *D* the shear decreases, and the point *C* where the shear is zero is obtained by dividing 1160, the ordinate at *B*, by 200, the amount the shear diagram drops per foot. Distance $BC = 1160 \div 200 = 5.8$ ft. In the remaining 4.2 ft., the shear diagram drops a further distance of $-4.2 \times 200 = -840$ lb., then comes back to zero at *D* by the amount of R_2.

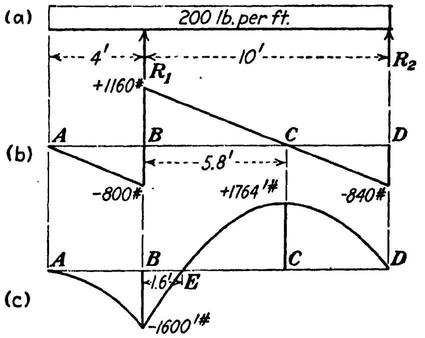


FIG. 83.

The points of maximum moment are points *B* and *C* where the shear diagram crosses the axis, at 4 ft. and at 9.8 ft. from the left end. At point *B*, the moment is given by the area of the shear diagram up to that point.

$$M_B = -\tfrac{1}{2}(800 \times 4) = -1600 \text{ ft.-lb.}$$

At point *C*, the moment is given by the algebraic sum of the area of the shear diagram from *A* to *C*.

$$M_C = -1600 + \tfrac{1}{2}(1160 \times 5.8) = +1764 \text{ ft.-lb.}$$

The larger numerical value, +1764 ft.-lb., is the controlling value of the maximum moment.

Point *E* where the moment is zero is located by equating the expression for the moment to zero.

$$M = 1960(x - 4) - 100x^2 = 0.$$
$$x = 5.6 \text{ ft.}$$

The moment diagram is drawn in Fig. 83(*c*). It will be seen from the moment diagram that the moment at point *C* is a mathematical maximum.

The value of x at which this maximum occurs may therefore be obtained by equating to zero the first derivative of the moment with respect to the length.

$$\frac{dM}{dx} = 1960 - 200x = 0.$$
$$x = 9.8 \text{ ft.}$$

The maximum value of the moment occurring at point B is not a mathematical maximum, since the slope of the moment curve changes suddenly from negative to positive and does not pass through zero. The value of x to give this maximum cannot be obtained by equating to zero the first derivative of the moment with respect to the length. It is seen, however, that in each case the maximum moment occurs at the point where the shear diagram crosses the axis.

PROBLEMS

1. A cantilever beam 80 in. long carries a load of 600 lb. at the free end. Compute the maximum moment due to the 600-lb. load.
Ans. −48,000 in.-lb.

2. A cantilever beam 12 ft. long weighing 120 lb. per linear ft. carries a superimposed load of 600 lb. per linear ft. Compute the maximum moment.
Ans. −51,840 ft.-lb.

3. A cantilever beam weighs 60 lb. per linear ft. What length may it be made in order to cause a moment of 12,000 ft.-lb. at the fixed end?
Ans. 20 ft.

4. A simple beam 30 ft. long carries a load of 16,000 lb. at the middle. Compute the maximum moment due to the concentrated load.
Ans. 120,000 ft.-lb.

5. A simple beam 100 ft. long carries a uniformly distributed load of 3000 lb. per linear ft. and a concentrated load of 100,000 lb. at the middle. Compute the maximum moment. *Ans.* 75,000,000 in.-lb.

6. A beam 30 ft. long is supported at points 5 ft. from each end and carries a uniformly distributed load of 400 lb. per linear ft. Compute the maximum moment. *Ans.* 15,000 ft.-lb.

7. Solve Problem 6 if a concentrated load of 1500 lb. is added at each end.
Ans. 12,500 ft.-lb.

43. Maximum Moment Due to Moving Loads.—If a system of concentrated loads with fixed distances between the loads moves across a simple beam, the moment under each load in turn increases from zero up to a maximum value and then reduces to zero again as that load moves across the beam. It is required to find the location of any given load, as P_2, Fig. 84, when the moment under it is a maximum. Let l be the length of the beam, x the distance of P_2 from the left end of the beam, and a the distance from P_2 to the position of the resultant ΣP of all of the moving loads on the beam. From the equation of moments about R_2, $R_1 l = \Sigma P(l - x - a)$. The moment with respect to P_2 of all of the forces on the left is

$$M = R_1 x - P_1 b,$$
$$= \frac{\Sigma P}{l}(l - x - a)x - P_1 b.$$

To determine the value of x which will give a maximum value of M, the first derivative of M with respect to x is equated to zero.

$$\frac{dM}{dx} = \frac{\Sigma P}{l}(l - 2x - a) = 0.$$
$$l - 2x - a = 0.$$
$$x = \frac{l}{2} - \frac{a}{2}.$$

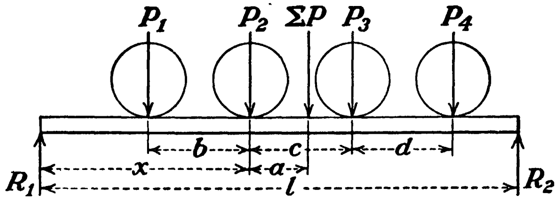


FIG. 84.

When $x = \frac{l}{2} - \frac{a}{2}$, the center of the beam is midway between P_2 and ΣP.

The moment under any one of a given system of moving loads is a maximum when the center of the beam bisects the distance between the given load and ΣP; that is, the given wheel is as far one way from the center of the beam as the center of gravity of the moving loads is the other way.

It is usually necessary to examine only for the maximum moments under the two loads nearest ΣP. If the largest load of all on the beam is nearest ΣP, it will be necessary to examine only under it.

EXAMPLE

In Fig. 84, let $l = 50$ ft., $b = 9$ ft., $c = d = 7$ ft., $P_1 = 20{,}000$ lb., $P_2 = P_3 = P_4 = 50{,}000$ lb. Compute the maximum moment under P_2 and under P_3.

Solution.—By moments of the four loads about P_2, $a = 5.12$ ft. The moment under P_2 is a maximum when it is at a point 2.56 ft. on the left of the middle of the beam, so $x = 22.44$ ft. $R_1 = 76{,}300$ lb. The moment under P_2 is

$$M = 76{,}300 \times 22.44 - 20{,}000 \times 9 = 1{,}532{,}000 \text{ ft.-lb.}$$

The distance from P_3 to ΣP is 1.88 ft. The moment under P_3 is a maximum when it is at a point 0.94 ft. on the right of the middle of the beam. For this position of the loads, $R_1 = 88{,}200$ lb. The moment under P_3 is

$$M = 88{,}200 \times 25.94 - 20{,}000 \times 16 - 50{,}000 \times 7 = 1{,}617{,}900 \text{ ft.-lb.}$$

PROBLEMS

1. In the Example above, compute the maximum moment under P_1 and under P_4. *Ans.* 1,094,300 ft.-lb.; 1,437,200 ft.-lb.

2. The weight on the front wheel of a road roller is 3 tons and that on the rear wheels is 10 tons. The wheel base is 10 ft. Compute the maximum moment on a 30-ft. girder as the roller moves across it. *Ans.* 166,200 ft.-lb.

3. The loads carried by the wheels of an articulated truck are as follows: steering wheels, 4000 lb.; driving wheels, 8000 lb.; rear wheels, 6000 lb. The distance from the steering wheels to the driving wheels is 10 ft. and that from the driving wheels to the rear wheels is 16 ft. Compute the maximum moment on a 40-ft. girder. *Ans.* 113,100 ft.-lb.

CHAPTER VI

STRESSES IN BEAMS

44. Distribution of Tensile and Compressive Stresses.—It is found by experiment that a beam which is originally straight and which is bent by transverse loads and reactions is elongated on the convex side and shortened on the concave side, while at a surface between the two there is no change of length. This surface where fibers do not change in length is called the *neutral surface* of the beam, and its intersection with any cross section is called the *neutral axis* of the cross section.

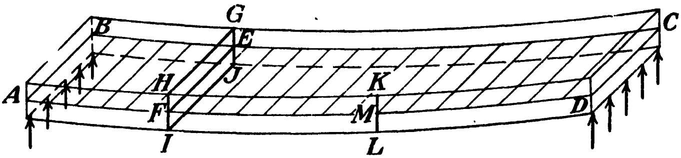


Fig. 85.

In Fig. 85, *ABCD* is the neutral surface of the beam shown, and *EF* is the neutral axis of the cross section *GHIJ*.

It is also found by experiment that two transverse parallel straight lines, such as *HI* and *KL* inscribed on the side of the beam before it is loaded, remain straight after the beam is loaded, provided the elastic limit of the material is not exceeded. The length *FM* along this neutral surface remains unchanged while *HK* is shortened and *IL* is lengthened.

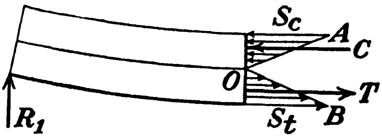
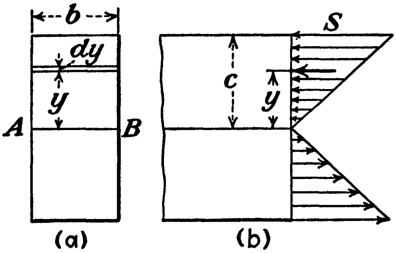


Fig. 86.

Since stress is proportional to deformation, it follows that the stress in the beam must vary uniformly from zero at the neutral surface to a maximum value at the outer fiber. In Fig. 86, vector *C* represents the resultant internal compressive stress at the section acting on the left part of the beam as the free body. Vector *T* represents the resultant internal tensile stress. Vectors S_c and S_t represent, respectively, the unit compressive and tensile stresses at the outer fiber, while the ordinates to the straight lines *OA* and *OB* represent to the same scale the unit stresses at the corresponding points in the cross section.

45. Relation between Moment and Internal Stress.—As has already been shown in Art. 37, the resultant tensile and compressive forces *T* and *C* in Fig. 86 are equal, and therefore constitute a couple, the moment of which balances that of the external forces acting upon the beam on the left of the section. Figure 87(*a*) shows the cross section of a beam, and Fig. 87(*b*) shows the side view of a small part of the beam just on the left of the cross section. (The section shown is rectangular but the following discussion is valid for a cross section of any shape.)



(a) (b)

FIG. 87.

The line *AB* is the neutral axis of the cross section, *b* is the width of the beam, *y* is the distance from the neutral axis to any differential area $dA = bdy$. Let *c* be the distance from the neutral axis to the outer fiber.

S is the unit stress at the outer fiber.

$\frac{S}{c}$ is the unit stress at unit distance from the neutral axis.

$\frac{S}{c}y$ is the unit stress at the distance *y* from the neutral axis.

$\frac{S}{c}ydA$ is the force on the differential area *dA*.

$\frac{S}{c}y^2dA$ is the moment of the force on area *dA* with respect to axis *AB*.

$\frac{S}{c}\int y^2dA$ is the total moment of all of the internal forces with respect to axis *AB*.

The quantity $\int y^2 dA$ is I, the moment of inertia of the cross section with respect to the X axis, so the expression for the internal moment becomes SI/c.

Since this internal moment holds in equilibrium the external moment M,

$$M = \frac{SI}{c}.$$

$$S = \frac{Mc}{I}.$$

The unit stress at any other point than the outer fiber is proportional to its distance from the neutral axis, and is given by the expression,

$$S = \frac{My}{I}.$$

46. Location of the Neutral Axis.—As was shown in Art. 45, $\frac{S}{c}ydA$ is the horizontal force on the differential area dA.

$\frac{S}{c}\int ydA$ is the total horizontal force on the section.

By Mechanics,

$$\int ydA = \bar{y}A,$$

A being the area of the cross section and $\bar{y}$ the distance from the axis from which y is measured, to the centroidal axis of the cross section. Since there are no other horizontal forces acting upon the free body, and since the summation of horizontal forces must equal zero,

$$\frac{S}{c}\bar{y}A \textit{ must equal zero.}$$

Since neither S nor A can equal zero and since c cannot equal infinity, it is necessary that the quantity $\bar{y}$ equal zero. If $\bar{y}$ is zero, the neutral axis and the centroidal axis coincide.

The neutral axis of any cross section is the centroidal axis.

PROBLEMS

1. In Fig. 91(*c*), the flange of the T-beam is 4 by ½ in. and the web is 5½ by ½ in. Locate the neutral axis from the base of the web.

Ans. 4.01 in.

2. The channels shown in Fig. 92(*d*) are 10-in. channels, each having an area of 5.86 sq. in. The distance from the back of the channel to the centroidal axis is 0.61 in., and the thickness of the web is 0.379 in. Locate the neutral axis from the base of the lower channel. *Ans.* 7.385 in.

47. Section Modulus.

—The flexure formula, $M = SI/c$, derived in Art. 45, may also be written in the form,

$$\frac{M}{S} = \frac{I}{c}.$$

The quantity I/c is called the *strength modulus of the section*, or more briefly, the *section modulus*. As its name indicates, it is a measure of the comparative strength of beams. If, for instance, the section modulus I/c of the cross section of one beam is twice as great as that of another of the same material, it is twice as strong.

The values of the section moduli of some structural shapes are given in Tables IX to XII at the end of this volume. Complete tables of the values of the section moduli for all rolled structural shapes, and for certain built-up sections, will be found in the handbooks of steel companies. If a section is unsymmetrical with respect to the centroidal axis, it will have two section moduli of different values, the lesser of which is the governing value.

For a solid rectangular section, the moment of inertia is $bd^3/12$ and the value of c is $d/2$, so the section modulus is $I/c = bd^2/6$. Since the unit of moment of inertia I is in.^{4}, and that of distance c is inches, the unit of the section modulus I/c is inches cubed.

PROBLEMS

1. Compute the section modulus of a beam 16 in. wide and 20 in. deep. *Ans.* 1067 in.^{3}.

2. The flanges of a timber beam with an I-section are 10 in. wide and 2 in. deep, and the web is 2 in. wide and 12 in. deep. Compute the section modulus. *Ans.* 282.7 in.^{3}.

3. Compute the section modulus of a pipe which is 4.5 in. outside diameter and 4.026 in. inside diameter. *Ans.* 3.21 in.^{3}.

4. The cross section of a triangular beam has a base of 2 in. and an altitude of 3 in. Compute the two values of the section modulus. *Ans.* 0.75 in.^{3}; 1.5 in.^{3}.

48. Computation of Flexural Stresses.

—In the examination of beams, the flexural stress usually desired is that at the outer fiber at the section in the beam where the moment is a maximum. This is the maximum value of the flexural stress in the beam, and is the governing value except in special cases. The chief exception is the beam of unsymmetrical section composed of material which has an allowable stress in tension different from that in compression. In this case a value of the stress less than the maximum may be the governing value.

In computing the maximum flexural stress in a beam, the location of the section of maximum moment is obtained as in Art. 42 and the amount of the maximum moment is computed. The value of the section modulus is then computed or obtained from tables of properties of sections. The expression $S = Mc/I$ derived above may also be used in the form,

$$S = \frac{M}{\frac{I}{c}}.$$

The maximum unit fiber stress at any section in a beam is obtained by dividing the moment at that section by the section modulus of the cross section.

It is necessary that the equation shall be homogeneous. In order that the value of the unit stress shall be given in pounds per square inch, the moment must be used in units of inch-pounds and the section modulus in units of inches to the third power.

EXAMPLE 1

Compute the maximum unit stress in a cantilever beam 6 in. wide, 10 in. deep, and 8 ft. long, when loaded with a uniformly distributed load of 250 lb. per linear ft.

Solution:

$$W = 250 \times 8 = 2000 \text{ lb.}$$
$$M = 2000 \times 48 = 96{,}000 \text{ in.-lb.}$$
$$\frac{I}{c} = \frac{bd^2}{6} = 100 \text{ in.}^3.$$
$$S = \frac{96{,}000}{100} = 960 \text{ lb./in.}^2.$$

EXAMPLE 2

A steel pipe 20 ft. long, 5 in. outside diameter, and 4.5 in. inside diameter, is supported at the ends and carries a concentrated load of 1000 lb. at the middle. Compute the maximum fiber stress due to this load.

Solution.—The maximum moment is at the middle and its amount is

$$M = \frac{Pl}{4} = 60{,}000 \text{ in.-lb.}$$

$$\frac{I}{c} = \frac{\pi}{4r}(r^4 - r_1^4) = \frac{\pi}{10}(39.06 - 25.62) = 4.224 \text{ in.}^3.$$

$$S = \frac{60{,}000}{4.224} = 14{,}200 \text{ lb./in.}^2.$$

EXAMPLE 3

A 24-in. 90-lb. I-beam 40 ft. long, supported at the ends, carries an additional uniformly distributed load of 800 lb. per linear ft. and a concentrated load of 10,000 lb. at the middle. Compute the maximum stress.

Solution.—The total uniformly distributed load is $W = 890 \times 40 =$ 35,600 lb. The maximum moment is at the middle and is $M = \frac{Wl}{8} + \frac{Pl}{4}$.

$$M = 35{,}600 \times 40 \times \frac{12}{8} + 10{,}000 \times 40 \times \frac{12}{4}.$$

$$M = 3{,}336{,}000 \text{ in.-lb.}$$

From Table IX, the section modulus $I/c = 185.8$ in.^{3}.

$$S = \frac{3{,}336{,}000}{185.8} = 17{,}950 \text{ lb./in.}^2.$$

EXAMPLE 4

A 5-in. 10-lb. I-beam 20 ft. long is supported at the right end and at a point 4 ft. from the left end. In addition to its own weight, it carries a uniformly distributed load of 100 lb. per linear ft., a concentrated load of 1000 lb. at the left end, and another of 3000 lb. at a point 12 ft. from the left end. Compute the maximum stress.

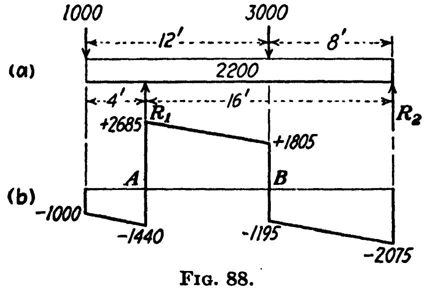


FIG. 88.

Solution.—To determine R_1, the moment equation with respect to R_2 is written.

$$16R_1 = 1000 \times 20 + 3000 \times 8 + 2200 \times 10.$$

$$R_1 = 4125 \text{ lb.}$$

$$R_2 = 6200 - 4125 = 2075 \text{ lb.}$$

For this beam it is easier to draw the shear diagram to determine the position and amount of the maximum moment. The beam is shown drawn to scale in Fig. 88(*a*). With the values just obtained for the reactions, the shear diagram is drawn, as shown in Fig. 88(*b*). From this diagram it is seen that the two sections at which the moment is a maximum are *A* and *B*. The moment at *B* is plainly the larger, and its amount is given by the area of the trapezoid on the right of *B*.

$$M = \frac{(1195 + 2075)}{2} \times 8 \times 12 = 156{,}960 \text{ in.-lb.}$$

From Table IX,

$$\frac{I}{c} = 4.8 \text{ in.}^3$$

$$S = \frac{156{,}960}{4.8} = 32{,}700 \text{ lb./in.}^2$$

This value is below the elastic limit for structural steel, so the computation is valid. It should be noted, however, that this is considerably higher than the value commonly used.

EXAMPLE 5

A cast-iron T-beam with a cross section as shown in Fig. 89 projects 5 ft. from a wall as a cantilever beam and carries a uniformly distributed load of 400 lb. per linear ft. including its own weight. Compute the maximum unit tensile stress and the maximum unit compressive stress.

Solution.—The distance of the centroidal axis of the cross section from the upper edge of the section is given by the equation,

$$10\bar{y} = 6 \times 0.5 + 4 \times 3.$$
$$\bar{y} = 1.5 \text{ in.}$$

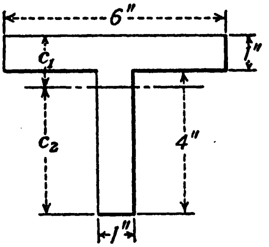


FIG. 89.

The distance c_1 from the neutral axis to the top fiber is therefore 1.5 in., and the distance c_2 from the neutral axis to the bottom fiber is 3.5 in. $I = 20.83$ in.^{4}. The maximum moment is at the fixed end and is

$$M = 2000 \times 2.5 \times 12 = 60{,}000 \text{ in.-lb.}$$

At the top fiber,

$$S_t = 60{,}000 \times \frac{1.5}{20.83} = 4320 \text{ lb./in.}^2$$

At the bottom fiber,

$$S_c = 60{,}000 \times \frac{3.5}{20.83} = 10{,}080 \text{ lb./in.}^2$$

The New York Building Laws (Table III) give 3000 lb./in.^{2} in tension and 16,000 lb./in.^{2} in compression as the allowable values. According to these values, the stress in tension is the governing value, and is higher than is allowed, while the stress in compression is well below the allowable value.

PROBLEMS

1. A timber cantilever beam 10 in. wide, 3 in. deep, and 6 ft. long carries a load of 200 lb. at the free end. Compute the maximum flexural stress due to the load. *Ans.* 960 lb./in.^{2}.

2. If the timber beam described in Problem 1 weighs 36 lb./cu. ft., what is the maximum flexural stress due to its own weight? *Ans.* 108 lb./in.^{2}.

3. A simple cast-iron beam is 2 in. wide, 1.5 in. deep, and 6 in. long between supports. Compute the load at the middle which will cause a stress of 4000 lb./in.^{2} in bending. *Ans.* 2000 lb.

4. A timber girder 8 in. wide, 18 in. deep, and 24 ft. long is supported at the ends and carries a uniformly distributed load of 400 lb. per linear ft. including its own weight. Compute the maximum flexural stress. Compute also the additional concentrated load which will increase the maximum stress to 1400 lb./in.^{2}. *Ans.* 800 lb./in.^{2}; 3600 lb.

5. A box girder is made of two 2- by 12-in. planks and two 2- by 16-in. planks 20 ft. long bolted together to make it 12 in. wide and 20 in. deep outside dimensions. If the girder is supported at the ends and loaded with a uniformly distributed load of 1000 lb. per linear ft. including its own weight, what is the maximum flexural stress? What is the maximum stress in the vertical 16-in. planks? *Ans.* 1140 lb./in.^{2}; 910 lb./in.^{2}.

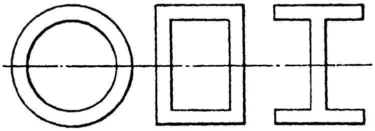
6. The horizontal cantilever boom of a crane is 10 ft. long and is composed of two 6-in. 8.2-lb. channels, with the webs vertical. What load at the free end will cause a maximum stress of 20,000 lb./in.^{2}? *Ans.* 1445 lb.

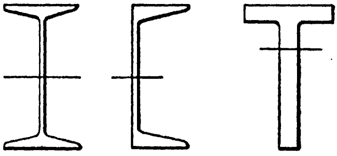
7. A standard 20-in. 100-lb. I-beam 35 ft. long is supported at the ends. Compute the load per linear foot which will cause a maximum flexural stress of 18,000 lb./in.^{2}. *Ans.* 1614 lb./ft.

8. A simple beam 60 ft. long is composed of two 36-in. 300-lb. Carnegie-beam sections and is loaded with a total uniformly distributed load of 420,000 lb. Compute the maximum flexural stress.

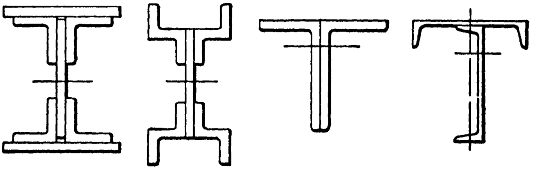
Ans. 17,100 lb./in.^{2}.

49. Economic Sections.—As was shown in Art. 45, the flexural fiber stress at any section in a beam varies from a maximum value at the outer fiber to zero at the neutral axis. It is therefore evident that in beams which have rectangular or circular cross sections there is a large amount of the material near the neutral axis which is stressed only slightly. If the cross section is modified by removing some of the material near the neutral surface and placing it farther away from the neutral surface, the section modulus will be made larger and hence the strength of the section will be increased. The hollow circular section, the hollow rectangular or box-girder section, and the I-section, shown in Fig. 90, are examples of such modified cross sections. Rolled structural steel I-beams, channels, and T-beams, Fig. 91, and various built-up sections such as shown in Fig. 92, are other examples of sections which have been modified so as to increase the section modulus, and therefore the strength, for a given weight of material.


(a) (b) (c)
FIG. 90.


(a) (b) (c)
FIG. 91.

The T-section is especially advantageous for cast-iron beams, since the outer fibers at the bottom of the stem are farther from the neutral surface than those at the top of the flange, and therefore have the higher stress. Such beams should always be loaded so as to cause tension in the flange, and compression in the stem of the section.


(a) (b) (c) (d)
FIG. 92.

PROBLEMS

1. If a shaft 3 in. in diameter is hollow-bored so as to make it $2\frac{1}{2}$ in. inside diameter, how much of the weight is eliminated and how much is the strength reduced? *Ans.* 69.5 per cent; 48.1 per cent.

2. Compare the strength of a beam 8 in. wide and 12 in. deep with that of a hollow box beam of the same weight which is 12 in. wide and 16 in. deep outside and 8 in. wide and 12 in. deep inside. *Ans.* 1 to 1.92.

50. Design of Beams.—In some cases the length of a beam, the manner of its support, the character and amount of its loading, and the allowable stress, are known, from which the dimensions of the cross section are to be computed. The amount of the maximum moment is computed and from the flexure formula $M/S = I/c$, the required numerical value of the section modulus is determined.

The value of I/c for a solid rectangular section is $bd^2/6$. In this case it is necessary to assume the value of either b or d, or to assume some relation between them. In the design of a wooden floor-beam system, some commercial size of joist is selected and the spacing is computed.

For a solid circular section the value of I/c is $\frac{1}{4}\pi r^3$, so the radius of the cross section may be computed directly.

EXAMPLE 1

A cantilever beam 2 ft. long is to support a load of 120 lb. at the free end. Neglecting the weight of the beam, compute the diameter of a solid circular steel beam to carry the load with an allowable stress of 12,000 lb./in.^{2}.

Solution.—The maximum moment is 120 × 24 = 2880 in.-lb.

$$\frac{M}{S} = \frac{2880}{12{,}000} = 0.24.$$

$$\frac{I}{c} = \frac{\pi r^3}{4} = 0.24.$$

$$r^3 = 0.306.$$

$$r = 0.674 \text{ in.}$$

$$d = 1.348 \text{ in.}$$

The nearest commercial size of rolled bar is $1\frac{3}{8}$ in.

EXAMPLE 2

A simple wooden beam 12 ft. long with a rectangular cross section is to support a uniformly distributed load of 300 lb. per linear ft., including its own weight, with an allowable flexural stress of 1200 lb./in.^{2}. Assuming the depth to be twice the width, compute the dimensions.

Solution.—The total weight W is 12 × 300 = 3600 lb. The maximum moment

$$M = \frac{Wl}{8} = 144 \times \frac{3600}{8} = 64{,}800 \text{ in.-lb.}$$

$$\frac{M}{S} = \frac{64{,}800}{1200} = 54.$$

$$\frac{I}{c} = 54.$$

If $b = d/2$, $I/c = bd^2/6 = d^3/12$.

$$\frac{d^3}{12} = 54.$$

$$d^3 = 648.$$

$$d = 8.65 \text{ in.}$$

$$b = 4.325 \text{ in.}$$

EXAMPLE 3

A simple wooden beam 16 ft. long is to support a center load of 8000 lb. with an allowable stress of 1400 lb./in.^{2}. Assuming a depth of 12 in., compute the necessary width. Neglect the weight of the beam.

Solution:

$$M = \frac{Pl}{4} = 384{,}000 \text{ in.-lb.}$$

$$\frac{M}{S} = \frac{384{,}000}{1400} = 274.$$

If $d = 12$ in., $\frac{I}{c} = b \times \frac{144}{6} = 24b.$

$$24b = 274.$$
$$b = 11.4 \text{ in.}$$

EXAMPLE 4

Compute the spacing necessary for 2- by 14-in. joists 20 ft. long, supported at the ends, to carry a floor load of 100 lb./ft.^{2} of floor area, with an allowable flexural stress of 1200 lb./in.^{2}. The floor load is assumed to include the weight of the joists.

Solution.—Let x be the spacing of the joists, center to center, in feet. The load supported by one joist will be

$$W = 100 \times 20x.$$
$$M = 2000 \times 30x = 60{,}000x.$$
$$\frac{60{,}000x}{1200} = 2 \times \frac{14^2}{6}.$$
$$x = 1.307 \text{ ft.} = 15.68 \text{ in.}$$

PROBLEMS

1. A steel cantilever beam 40 in. long with a circular cross section, is to support an end load of 2000 lb. with an allowable flexural stress of 16,000 lb./in.^{2}. Compute the required diameter. *Ans.* 3.7 in.

2. By what percentage would the strength of the beam in Problem 1 be reduced by drilling a transverse hole ¼ in. in diameter through the middle of the beam at the fixed end? By drilling a transverse vertical hole of the same size? *Ans.* 0.052 per cent; 11.5 per cent.

3. A simple wooden beam 30 ft. long is to carry a uniformly distributed load of 600 lb. per linear ft. Compute the dimensions if the depth is to be twice the width and the allowable stress is 1400 lb./in.^{2}. *Ans.* 9.54 by 19.08 in.

4. A wooden cantilever beam 8 ft. long is to be made of 2- by 8-in. planks placed vertically and bolted together to carry a load of 800 lb. at the free end. Compute the number of planks necessary if the allowable stress is 1000 lb./in.^{2}. *Ans.* Four.

5. Compute the spacing of 2- by 16-in. joists 18 ft. long, supported at the ends, to carry a floor load of 120 lb./ft.^{2} of floor area, with an allowable stress of 1100 lb./in.^{2}. The assumed floor load includes the weight of the joists. *Ans.* 19.3 in.

51. Selection of Rolled Structural Beams.—Rolled structural steel shapes are often used as beams in buildings, bridges, and similar structures, and are usually I-beams, channels, or T-beams. In case the loading on one beam is known, the problem is usually to select the lightest beam that will carry the load with a given allowable stress. It may be, however, that a heavier and shallower beam may be preferred on account of the saving of head room. Since the beam selected must support its own weight also, another computation is usually necessary to determine if it can do so without exceeding the allowable stress.

In some cases of design, with a given loading on a floor area, the size of floor beams is assumed. The problem then is to determine the necessary spacing for the given allowable stress. In such cases the assumed load usually includes the weight of the floor beams, so no further computation is necessary.

EXAMPLE 1

Select an American Standard I-beam for a span of 50 ft. to carry a superimposed uniformly distributed load of 660 lb. per linear ft., with an allowable stress of 18,000 lb./in.^{2}.

Solution:

$$M = \frac{Wl}{8} = 33{,}000 \times \frac{600}{8} = 2{,}475{,}000 \text{ in.-lb.}$$

$$\frac{M}{S} = \frac{2{,}475{,}000}{18{,}000} = 137.5.$$

In Table IX it is found that the first I-beam section which has a value of I/c larger than 137.5 is the 20-in. 81.4-lb. beam for which $I/c = 146.6$.

The weight of the beam itself is 4070 lb., and the added moment is $\frac{1}{8} \times 4070 \times 600 = 305{,}250$ in.-lb. The added value of I/c is $305{,}250/18{,}000 = 17.0$, making a required value of 154.5 for I/c. The 20-in. 81.4-lb. beam is not heavy enough, neither is the 20-in. 85-lb. beam nor the 20-in. 90-lb. beam.

For the 20-in. 95-lb. beam, the added weight is 4750 lb., the added moment is 356,250 in.-lb., and the added value of I/c is 19.8, making the required value of $I/c = 157.3$. The value of I/c as given in the table is 160.0, so this beam is strong enough.

It will be noticed, however, that the 24-in. 79.9-lb. beam is a lighter beam, and still has a value of I/c larger than the required value. So unless the added 4 in. of depth would prevent its use, the lighter 24-in. beam would be selected. The saving in weight would be 15.1 lb. per linear ft., or 755 lb. on the beam.

EXAMPLE 2

Compute the spacing of 8-in. 18.4-lb. I-beams necessary to support a uniformly distributed load of 250 lb./ft.^{2} of floor area. The span is 20 ft. and the allowable stress is 18,000 lb./in.^{2}. The assumed weight includes the weight of the I-beams.

Solution.—Let x be the spacing in feet. The load on one beam is

$$W = 5000x.$$

$$M = \frac{Wl}{8} = 150{,}000x.$$

$$\frac{M}{S} = \frac{150{,}000x}{18{,}000} = 8.33x.$$

From Table IX, $I/c = 14.2$.

$$8.33x = 14.2.$$

$$x = 1.705 \text{ ft.} = 20.4 \text{ in.}$$

PROBLEMS

1. Select the lightest American Standard I-beam for a span of 20 ft. to carry a uniformly distributed load of 200 lb. per linear ft., not including its own weight, and a concentrated load of 6000 lb. at the middle, with an allowable stress of 16,000 lb./in.^{2}. *Ans.* 12-in. 31.8-lb. beam.

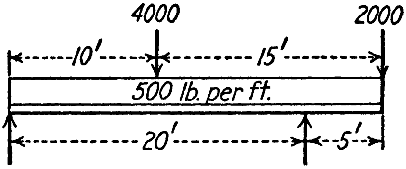


FIG. 93.

2. Select two channels, placed back to back, to support the same load as that given in Problem 1. *Ans.* Two 12-in. 20.7-lb. channels.

3. Select the lightest American Standard I-beam to carry the load shown in Fig. 93 with an allowable stress of 18,000 lb./in.^{2}. The load given does not include the weight of the beam. *Ans.* 10-in. 30-lb. beam.

4. Select the lightest American Standard I-beam to be used in a floor supporting a uniformly distributed load of 800 lb./ft.^{2} of floor area, including its own weight. The span is 25 ft., the spacing is 2 ft., and the allowable stress is 18,000 lb./in.^{2}. *Ans.* 18-in. 54.7-lb. beam.

5. Determine the necessary spacing for 5-in. 10-lb. I-beams to support a uniformly distributed load of 200 lb./ft.^{2} of floor area, including the weight of the beams. The span is 18 ft. and the allowable stress is 20,000 lb./in.^{2}. *Ans.* 11.85 in.

6. A cantilever beam 8 ft. long is to support a uniformly distributed load of 40 lb. per linear ft. including its own weight, and a concentrated load of

500 lb. at the free end, with an allowable stress of 17,500 lb./in.^{2}. Select two angles to be placed back to back to form a T-beam to carry the load.

Ans. Two 5- by 3- by $\frac{5}{16}$-in. angles, with 5-in. legs vertical.

52. Stresses above the Elastic Limit: Modulus of Rupture.—If the load on a beam is increased until the stress in the outer fibers exceeds the proportional elastic limit, the unit stress will no longer be proportional to the unit deformation, and the stress distribution will differ from that shown in Figs. 86 and 87. In case the stress-strain diagram for the material of the beam is the same in tension and compression, and the cross section is symmetrical with respect to the neutral axis, the neutral axis will remain coincident with the centroidal axis and the stress distribution will be as shown in Fig. 94. Lines OA and OC correspond to the straight-line part of the stress-strain diagram, and the lengths EA and FC represent the values of the elastic limit. The unit stress on the material near the neutral surface is greater than it would be if the stress were proportional to the deformation, while that at the outer fibers is less. The line OI shows the stress distribution necessary to furnish the same internal moment if the stress were proportional to the deformation through that range of stress.

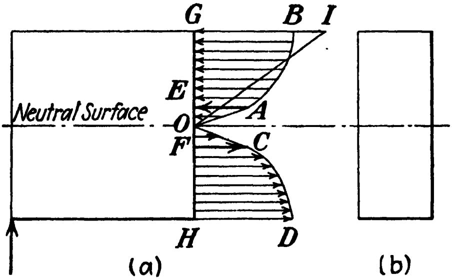


FIG. 94.

In case the cross section of the beam is not symmetrical with respect to the neutral axis, as the T-beam, Fig. 95, the fibers at E will be stressed higher than those at D. After the elastic limit has been passed at E and before it is passed at D, the unit stress diagram will take some such shape as indicated in Fig. 95(a). Since the fibers between F and E are not stressed proportionately and since the horizontal summation of forces must equal zero, the neutral axis is shifted upward away from the weaker part of the beam and no longer coincides with the centroidal axis. In Fig. 95(*b*), the area represents the total force, so area *GHI* must equal area *GKJ*.

In case the stress-strain diagram for the material of the beam is different in tension and in compression, the neutral surface will be shifted away from the part which has the lower stress, even though the section is symmetrical. For example, the proportional elastic limit for cast iron is much lower in tension than it is in compression. Figure 96 represents the stress distribution in a rectangular cast-iron beam which is loaded so that the upper fibers are stressed in compression and the lower fibers are stressed in tension. After the beam is loaded so that the unit stress at the outer fiber on the tension side exceeds the elastic limit, the neutral surface shifts from the center toward the compression side and reaches some such position as that at *O*. The material at *F* has reached its elastic limit in tension, line *AF* representing its amount, and the stress in the area below *F* exceeds the elastic limit. Above the neutral surface, the elastic limit in compression has not yet been reached, so the stress is proportional to the distance from the neutral surface.

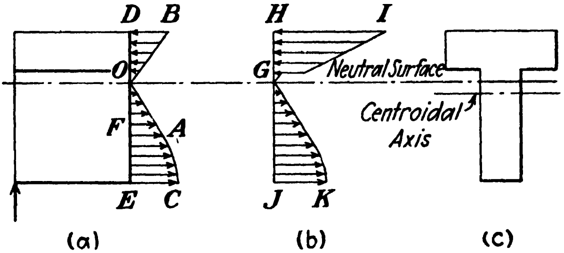


FIG. 95.

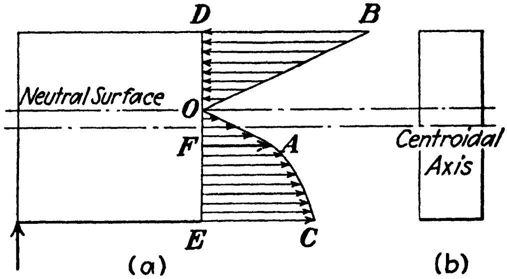


FIG. 96.

Since the total tensile force is necessarily equal to the total compressive force, the area $OACE$ must equal the area OBD.

It is evident that if the value of the elastic limit has been exceeded, neither the expression $S = My/I$ nor $S = Mc/I$ is valid, since the variation of stress is different from that assumed in their derivation. In the case of materials which rupture, however, it is customary to compute a value of S by means of the expression $S = Mc/I$, in which M is the value of the moment just preceding rupture. This value of S is not the true unit stress, but is called the *modulus of rupture,* or the measure of the rupture strength of the material. It is useful for the purpose of comparing the strengths of beams of the same material and shape.

Beams of ductile material will not rupture in direct flexure and therefore have no modulus of rupture.

PROBLEMS

1. A cast-iron beam 1.8 in. wide, 1.3 in. deep, and 8 in. long, supported at the ends, failed under a load of 7580 lb. applied at the middle. Compute the modulus of rupture. *Ans.* 29,800 lb./in.^{2}.

2. If the modulus of rupture of glass is 3600 lb./in.^{2}, what is the probable central breaking load for a plate of glass 10 in. wide, $\frac{3}{8}$ in. thick, and 4 ft. long, supported at the ends? *Ans.* 70.3 lb.

3. A wooden beam 7.75 in. wide, 15.75 in. deep, and 12 ft. long was supported at the ends and loaded with equal loads at the third points. The beam broke when each load was 103,600 lb. Compute the modulus of rupture, neglecting the weight of the beam. *Ans.* 15,500 lb./in.^{2}.

53. Stresses in Timber Beams: Form Factors.[1]—For steel, wrought iron, and similar materials, the value of the elastic limit in direct tension is the same as that in direct compression and agrees very closely with the value obtained from the flexure of beams. For cast iron, brick, stone, and similar materials, the value of the elastic limit in tension is much less than that in compression. As would be expected, the value of the elastic limit in flexure agrees with that in tension.

For timber, the value of the elastic limit in compression is much less than that in tension, and it might be expected that the value in flexure would agree with that in compression. This is not the case, however, but the value in flexure is from 25 to

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[1] See *Report* 181, National Advisory Committee for Aeronautics, by J. A. NEWLIN and G. W. TRAYER.

75 per cent higher. For example, an average of tests of Sitka spruce made by the Forest Products Laboratory, Madison, Wisconsin, in 1925, gives a value of 6500 pounds per square inch at elastic limit in flexure as compared with 4110 pounds per square inch in direct compression, a value 58 per cent higher. Newlin and Trayer account for this ability of the beam to take greater stress by the assumption that the minute wood fibers when subjected to longitudinal compression act as miniature long columns which give each other a certain amount of lateral support. If these miniature columns are all stressed alike as in direct compression, they cannot exert much lateral support on each other as they all bend together. If the stress is not uniform, however, as in a beam, the extreme fibers will be supported and stiffened by the less stressed fibers nearer the neutral surface and will therefore be able to support a higher stress. As a corroborative experiment, several matched pairs of test pieces 2 by 2 by 8 inches were tested parallel to the grain. One of each pair was loaded axially, the other with an eccentric load $\frac{1}{3}$ inch from the axis. Theoretically, the pieces loaded eccentrically should hold only one-half as much load as the others, the stress on the side nearest the load being a maximum and that on the opposite side being zero. The pieces were tested to failure, and instead of one-half as much load, they held over two-thirds as much load as those loaded axially. The fibers with less stress apparently furnished lateral support to those with greater stress and enabled them to carry a higher unit stress than that carried by the companion test pieces which were stressed equally over the entire cross section.

For purposes of comparison, a beam 2 by 2 inches in cross section was selected as the standard. The ratio of the strength of a beam of any other shape to that of the standard section is called the *form factor*. In a beam with an I-section such as is used in airplane wing beams, the supporting action of the less stressed fibers is limited to the depth of the flange except above the web, so a less value may be expected than that for the standard section. For a beam 5 inches deep with a web $\frac{13}{16}$ inch thick and flanges 1.5 inches wide and 0.5 inch deep, Newlin and Trayer obtained a value of the stress at the elastic limit which was 85 per cent of that for the 2- by 2-inch beam. Its form factor was therefore 0.85. The box beam acts similarly and for a section 3 by 7 inches outside and 2¼ by 4 inches inside, they obtained a value 73 per cent of that for the 2- by 2-inch beam, or a form factor of 0.73.

If the more highly stressed fibers are supported by a greater width of less stressed fibers, as is the case with the circular section and the square section with the diagonal vertical, the form factor is greater than unity. The square section with the diagonal vertical averaged as strong as the same section with the sides vertical, and since the value of I/c is only 0.707 as much, the form factor is 1/0.707 or 1.414. The circular section averaged as strong as a square section of the same area with the sides vertical, and since its value of I/c is 0.85 as much, the form factor is 1/0.85 or 1.18.

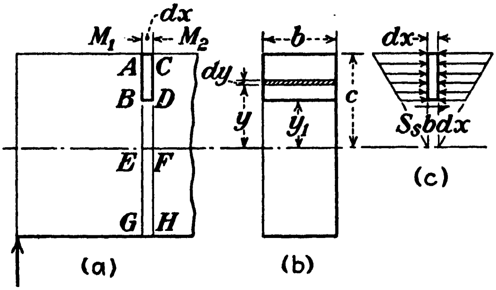


FIG. 97.

This supporting action of the fibers near the neutral surface upon those farther out still continues above the elastic limit. Newlin and Trayer found that the form factor at the modulus of rupture was slightly less than that at the elastic limit, the values for the two beams referred to above being 0.83 and 0.72, respectively. For eight types of box and I-beams tested, the form factor at the elastic limit averaged 0.78 while that at the modulus of rupture averaged only 0.72, a value 7.7 per cent less.

Tables XIII and XIV give the values of the modulus of rupture of a number of species of timber as determined by tests at the Forest Products Laboratory, U. S. Forest Service, Madison, Wisconsin.

54. Distribution of Shearing Stresses.—It has already been shown in Art. 16 that if a bar or a beam is subjected to the action of shearing forces in any given direction, a set of secondary shearing stresses of equal unit intensity is induced on planes normal to the direction of the original shearing forces.

Figure 97(*a*) represents a side view of a beam in which *AG* and *CH* are two cross sections of the beam *dx* apart. Figure 97(*b*) represents either of these cross sections. Figure 97(*c*) represents the differential block *ABDC* as a free body, with the horizontal forces shown. (The vertical forces are omitted in the figure.) Upon the left face, where the moment is M_1, the variable unit stress is $S_1 = M_1y/I$. The force upon the differential area bdy or dA is $(M_1y/I)dA$ and the total horizontal force upon the left side of the differential block is given by

$$\frac{M_1}{I}\int_{y_1}^{c} ydA.$$

Similarly, the total horizontal force acting to the left upon the right side of the block is given by

$$\frac{M_2}{I}\int_{y_1}^{c} ydA.$$

The unit shearing force upon the lower surface of the block is S_s, and the total horizontal shearing force is S_sbdx. Since the free body is in equilibrium, the summation of the horizontal forces is equal to zero, or

$$\frac{M_2}{I}\int_{y_1}^{c} ydA - \frac{M_1}{I}\int_{y_1}^{c} ydA - S_sbdx = 0.$$

$$S_s = \frac{M_2 - M_1}{Ibdx}\int_{y_1}^{c} ydA.$$

The quantity $M_2 - M_1$ is dM, since it is the difference in the moment in length dx, and $dM/dx = V$, as was shown in Art. 41. The integral expression $\int_{y_1}^{c} ydA$ is equal to $\bar{y}A'$, A' being the area of the cross section above the plane upon which the horizontal shearing stress is being determined, and $\bar{y}$ the distance from the neutral axis to the center of gravity of the area A'. Then

$$S_s = \frac{V}{Ib}\bar{y}A'.$$

In a rectangular section of width b and depth d, the integral expression becomes

$$S_s = \frac{V}{Ib}\int_{y_1}^{c} bydy.$$

$$S_s = \frac{V}{2I}(c^2 - y_1^2).$$

This is the equation of a parabola, as shown in Fig. 98. The value of S_s is zero when $y_1 = c$ and is a maximum at the neutral surface where $y_1 = 0$. The amount of this maximum stress is

$$S_s = \frac{Vc^2}{2I} = \frac{3}{2}\frac{V}{bd}.$$

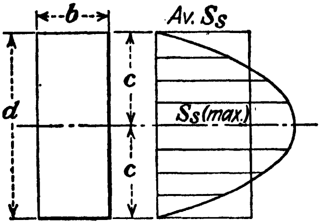


FIG. 98.

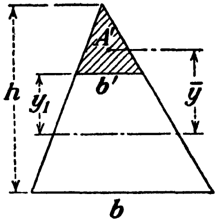


FIG. 99.

Since the quantity V/bd is the average unit shearing stress, it is seen that the maximum horizontal unit shearing stress in a beam of rectangular cross section is equal to 1.5 times the average vertical unit shearing stress. By Art. 16, the intensity of the vertical shearing stress varies in the same way as does the horizontal shearing stress.

If the cross section of a beam is not symmetrical with respect to the centroidal axis, the maximum shearing stress may not be at the neutral surface. In this case, the expression $S_s = (V/Ib)\bar{y}A'$ must be expressed in terms of y_1, the distance from the centroidal axis to the base of area A'. If this expression is then differentiated and the first derivative of S_s with respect to y_1 equated to zero, the value of y_1 to give a maximum value of S_s is obtained.

For the triangular cross section shown in Fig. 99, the expression for S_s becomes

$$S_s = \frac{V}{Ib'}\left[y_1 + \frac{1}{3}\left(\frac{2}{3}h - y_1\right)\right]\left[\frac{b'}{2}\left(\frac{2}{3}h - y_1\right)\right]$$

$$S_s = \frac{V}{I}\left(\frac{hy_1}{9} - \frac{y_1^2}{3} + \frac{2}{27}h^2\right).$$

$$\frac{dS_s}{dy_1} = \frac{V}{I}\left(\frac{h}{9} - \frac{2}{3}y_1\right).$$

For a maximum value of S_s,

$$\frac{h}{9} - \frac{2}{3}y_1 = 0.$$

$$y_1 = \frac{h}{6}.$$

It will be seen that in this case the maximum value of S_s is midway between the top and bottom of the beam.

Tables III, IV, and V give the allowable values of some materials in horizontal shear.

PROBLEMS

1. A timber beam 6 in. wide, 20 in. deep, and 10 ft. long, supported at the ends, carries a concentrated load of 16,000 lb. at the middle. Compute the unit horizontal shearing stress at points 2, 4, 6, 8, and 10 in. from the top of the beam.

Ans. 36 lb./in.^{2}; 64 lb./in.^{2}; 84 lb./in.^{2}; 96 lb./in.^{2}; 100 lb./in.^{2}.

2. Using the allowable value of S_s for southern yellow pine given in Table V, compute the total load uniformly distributed which can be placed on a beam 8 in. wide, 12 in. deep, and 6 ft. long, supported at the ends. Compute the maximum flexural stress when so loaded.

Ans. W = 14,080 lb.; 660 lb./in.^{2}.

3. For a southern yellow pine beam 6 in. wide and 18 in. deep, compute the length above which flexure governs and below which shear governs. Use values given in Table V. *Ans.* 21 ft. 10 in.

4. A beam 4 in. wide, 8 in. deep, and 8 ft. long, supported at the ends, is made by bolting together two beams each 4 in. square. If the bolts are spaced 3 in. apart in one row, and the beam is loaded with a load in the middle to cause a maximum flexural stress of 1500 lb./in.^{2}, what is the total shearing force on each bolt? *Ans.* 750 lb.

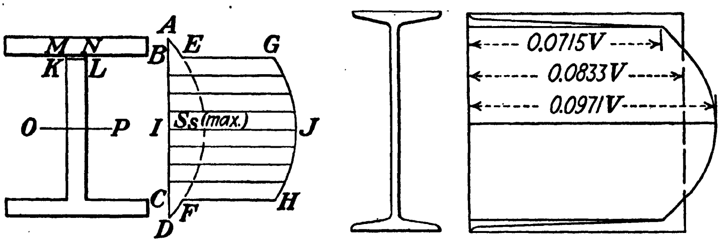
5. Three timbers, each 10 in. wide, 4 in. deep, and 16 ft. long, supported at the ends, are bolted together with $\frac{5}{8}$-in. bolts placed in two rows 6 in. apart in the row, to form a beam 10 in. wide and 12 in. deep. Compute the unit shearing stress in each bolt at the section between the timbers when the beam is loaded with a concentrated load at the middle to cause a maximum flexural stress of 1600 lb./in.^{2}. *Ans.* 12,090 lb./in.^{2}.

6. If the middle timber of the beam described in Problem 5 is placed with its 10 in. dimension vertical, so that the three timbers form an I-section, what must be the spacing of ½-in. bolts in a single row to carry the same load with an allowable shearing stress of 10,000 lb./in.^{2} in the bolts?

Ans. 7.4 in.

7. A timber beam 8 in. wide, 18 in. deep, and 25 ft. long is supported and loaded as shown in Fig. 93. Compute the maximum unit horizontal shearing stress due to the superimposed loads. *Ans.* 81.5 lb./in.^{2}.

55. Shearing Stresses in Structural I-Beams and Channels.—In beams with I-sections such as that described in Problem 6, Art. 54, the variation in the value of the horizontal unit shearing stress is as shown in Fig. 100(*a*). At the top and bottom fibers the stress is zero. At the lower edge of the upper flange the unit stress would be represented by the vector *BE* if the stress were of uniform intensity in the horizontal plane. The intensity at the middle in this plane, that is at *MN*, is greater than the average and approaches the value represented by the vector *BG*, which is the unit intensity of the shearing stress in the upper plane in the web, at *KL*.



(a) (b)

FIG. 100.

The shearing stress is a maximum at the neutral surface *OP* and its value is represented by the ordinate *IJ*. Both of the curves, *AEFD* and *GJH*, are parabolic.

If the section is the rolled I-section of the standard steel I-beam, in which the thickness of the flange increases toward the web, the shearing stress diagram is further modified, as shown in Fig. 100(*b*). In the rectangular part of the flange and in the web the variation is the same as in Fig. 100(*a*). In the triangular part of the flange the unit shearing stress varies gradually from the low stress in the rectangular part of the flange to the larger stress in the web.

It is a common commercial approximation to consider that the web, extending entirely through the flange, carries all of the shear, uniformly distributed, and that the remainder of the flange carries none. Figure 100(*b*) shows a comparison of the values obtained by the two methods for a 24-inch 79.9-pound I-beam. By accurate computation, the unit shearing stress at the base of the flange is $0.0715V$, V being the total vertical shear. The maximum unit shearing stress is at the neutral axis and its amount is $0.0971V$. The approximate value as commonly computed is $0.0833V$, which is intermediate between the value at the base of the flange and that at the neutral surface.

The variation in the unit horizontal shearing stress in standard rolled channel sections is similar to that in I-beams. The web of a channel is usually thicker in proportion to the flange area than is that of an I-beam, so the unit horizontal shearing stress is less for similar loading.

Problems

1. A 15-in. 50-lb. I-beam 10 ft. long, supported at the ends, is loaded with a concentrated load at the middle to cause a maximum stress of 18,000 lb./in.^{2}. Compute the unit stress at the base of the flange (where the fillet begins) and at the neutral surface. Neglect the fillets and rounded corners. Compute also the average unit shearing stress by the approximate method. *Ans.* 2000 lb./in.^{2}; 2780 lb./in.^{2}; 2330 lb./in.^{2}.

2. Solve Problem 1 for a 15-in. 50-lb. channel.
Ans. 1140 lb./in.^{2}; 1910 lb./in.^{2}; 1500 lb./in.^{2}.

3. If the maximum allowable horizontal shearing stress in the web of the channel described in Problem 2 is 15,000 lb./in.^{2}, what is the limiting load that can be placed upon a channel, supported at the ends, and of short span? *Ans.* 252,000 lb.

4. If allowable flexural stress S = 18,000 lb./in.^{2} and allowable horizontal shearing stress S_s = 12,000 lb./in.^{2} for a 15-in. 50-lb. I-beam, determine the length of span above which flexure governs and below which shear governs for a uniformly distributed load. *Ans.* 4.65 ft.

5. Solve Problem 4 by the commercial approximation of using S_s = 12,000 lb./in.^{2} as the allowable average vertical unit shearing stress on the gross area of the web. *Ans.* 3.9 ft.

56. Riveting and Welding of Plate Girders.—A plate girder consists of a vertical web plate A, Fig. 101, to which are riveted four structural angles B, B, and, if necessary, one or more flange plates C, C. As in the case of rolled I-beams and channels, the usual commercial approximation is to consider all of the vertical shear to be carried by the web plate with a uniform stress distribution. In this case, however, the effective depth of the web is the distance h between the center lines of the web rivets. If t is the thickness of the web, h the distance between the upper and lower lines of rivets, and V the total vertical shear, the average unit vertical shearing stress at the section MN is

$$\text{Ave. } S_s = \frac{V}{ht}.$$

The unit intensity of the horizontal shearing stress is the same in amount as the unit vertical shearing stress, so, if the slight reduction in the vertical shear between M and P is neglected, the rivets in the length $h_1 = h$ must transmit an amount of shear equal to V from the flange angles to the web plate. If n is the number of rivets in the length h_1 and R is the working strength of one rivet,

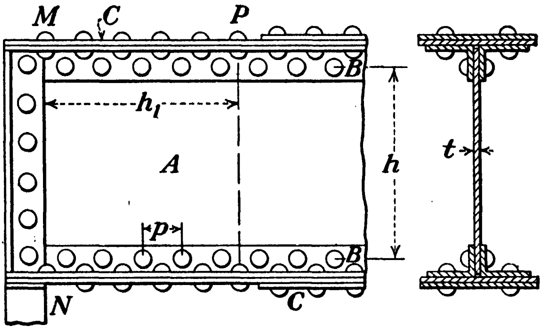


FIG. 101.

$$nR = V,$$

or

$$n = \frac{V}{R}.$$

The value of p, the pitch of the rivets, is given by

$$p = \frac{h_1}{n} = \frac{h_1 R}{V}.$$

Each rivet between the flange angles and the web plate is subjected to a shearing force at two areas, so its working strength in shear is given by the expression,

$$R = \frac{2\pi d^2 S_s}{4},$$

in which d is the diameter of the rivet and S_s is the allowable unit shearing stress in the rivet.

Each rivet is also subjected to side compression on an area td, so its working strength in compression is given by the expression,

$$R = tdS_c,$$

in which S_c is the allowable unit bearing or compressive stress on the side of the rivet. The smaller of the two values of R is the one to be used.

At the vertical cross section at P the total vertical shear V is less than that at M by the amount of the load between M and P. With this new value of V, the necessary pitch of the rivets in the second length $h_1 = h$ on the right of P is computed, and so on to the middle of the girder. If the girder carries only uniformly distributed load, the shear near the middle of the girder is small and the computed pitch of the rivets becomes large, but by the usual construction specifications the pitch is not permitted to exceed a certain maximum value.

The shearing force transmitted from the flange plates to the flange angles is less than that transmitted from the flange angles to the web plate, but the pitch of the rivets is usually made the same. Each rivet is subjected to shear at only one section, but there are twice as many rivets, so the total shearing resistance is the same. As in structural riveting, the nominal diameter of the rivet is used.

Vertical stiffener angles are riveted to the web where necessary, as will be explained in Art. 126.

Plate girders may be fabricated by welding the flange plates directly to the web plate by means of fillet welds. Since there are no rivet holes in the tension area, the total flange area required is reduced by that amount. In this case the double fillet weld in length $h_1 = h$ must have a total strength equal to the vertical shear V at the left of the section. For a plate girder with a uniformly distributed load, the shear V toward the middle of the girder becomes very small, so only a small amount of fillet weld would be necessary, but both for looks and for protection of the metal, the welding should be uniform throughout.

Outstanding vertical stiffener plates are used instead of stiffener angles, thus reducing the weight still further. These also are welded to the web plate by fillet welds.

EXAMPLE

A plate girder 40 ft. long is composed of a web plate 48 by ⅜ in., four flange angles each 6 by 4 by ¾ in., spaced 48½ in. back to back, and two flange plates each 16 by ¾ in. The girder is loaded with a uniformly distributed load of 100,000 lb. and a concentrated load of 140,000 lb. at the middle. With a distance of 44 in. between rivet lines, and using ⅞-in. rivets, with allowable $S_s = 13{,}500$ lb./in.², and allowable $S_c = 30{,}000$ lb./in.², compute the pitch of the rivets in each 44-in. panel.

Solution.—The strength of one rivet in shear is given by the expression

$$R = \frac{2\pi d^2}{4} S_s = \frac{2\pi 49 \times 13{,}500}{4 \times 64} = 16{,}250 \text{ lb.}$$

The strength of one rivet in bearing is given by the expression

$$R = tdS_c = \tfrac{3}{8} \times \tfrac{7}{8} \times 30{,}000 = 9850 \text{ lb.}$$

The smaller value, 9850 lb., must be used. At the end where $V = 120{,}000$ lb.,

$$p_1 = \frac{h_1 R}{V} = \frac{44 \times 9850}{120{,}000} = 3.61 \text{ in.}$$

It is customary in plate-girder work to use the pitch to the nearest ¼ in., so use $p_1 = 3\tfrac{1}{2}$ in.

At the next panel point, 44 in. from the end, the vertical shear V is decreased by the weight of the uniform load in the first panel, and is

$$V = 120{,}000 - 9170 = 110{,}830 \text{ lb.}$$

In this panel,

$$p_2 = \frac{h_1 R}{V} = \frac{44 \times 9850}{110{,}830} = 3.92 \text{ in.} \qquad \text{Use } p_2 = 4 \text{ in.}$$

In the next four panels, the computations are as follows:

$$p_3 = \frac{44 \times 9850}{101{,}670} = 4.27 \text{ in.} \qquad \text{Use } p_3 = 4\tfrac{1}{4} \text{ in.}$$

$$p_4 = \frac{44 \times 9850}{92{,}500} = 4.69 \text{ in.} \qquad \text{Use } p_4 = 4\tfrac{3}{4} \text{ in.}$$

$$p_5 = \frac{44 \times 9850}{83{,}330} = 5.21 \text{ in.} \qquad \text{Use } p_5 = 5\tfrac{1}{4} \text{ in.}$$

$$p_6 = \frac{44 \times 9850}{74{,}170} = 5.85 \text{ in.} \qquad \text{Use } p_6 = 5\tfrac{3}{4} \text{ in.}$$

The last panel is only 20 in. wide. For equal-width panels, each panel would be 40 in. wide.

PROBLEMS

1. Compute the pitch of rivets for the plate girder described in the example above if ¾-in. rivets are used and the panels are 40 in. wide.

Ans. p_1 = 3 in.; p_2 = 3¼ in.; p_3 = 3½ in.; p_4 = 4 in.; p_5 = 4¼ in.; p_6 = 4¾ in.

2. A plate girder 36 ft. long, with a web plate 40 in. deep and ½ in. thick carries a total uniformly distributed load of 240,000 lb. The distance between lines of rivets is 36 in. Using ⅞-in. rivets for which the allowable value of S_s is 12,500 lb./in.^{2} and the allowable value of S_c is 25,000 lb./in.^{2}, compute the pitch of rivets for each 36-in. panel.

Ans. The computed values are: p_1 = 3.28 in.; p_2 = 3.94 in.; p_3 = 4.92 in.; p_4 = 6.56 in.; p_5 = 9.85 in.; p_6 = 19.70 in. (The specifications of the American Bridge Company give 6 in. as the maximum spacing for ⅞-in. rivets, so by these specifications p_4, p_5, and p_6 would all be 6 in.)

3. A plate girder 60 ft. long, with a web plate 76 in. deep and ½ in. thick, carries a uniformly distributed load of 12,000 lb. per linear ft. and a concentrated load of 300,000 lb. at the middle. The distance between rivet lines is 72 in. Using 1-in. rivets for which the allowable value of S_s is 13,500 lb./in.^{2} and the allowable value of S_c is 30,000 lb./in.^{2}, compute the required pitch of rivets in each 72-in. panel.

Ans. The computed values are p_1 = 2.12 in.; p_2 = 2.47 in.; p_3 = 2.95 in.; p_4 = 3.68 in.; p_5 = 4.86 in. (According to specifications, the minimum spacing should not be less than three times the diameter of the rivets, so in the first three panels the rivets must be staggered.)

CHAPTER VII

DEFLECTION OF BEAMS

57. Curvature of a Beam: Simple Flexure.—When a beam is loaded with transverse loads, causing tensile stresses on one side of the neutral surface and compressive stresses on the other side, the fibers subjected to tension are elongated and those subjected to compression are shortened. This causes the beam to curve and therefore to deflect from its original unstressed position.

Simple flexure is bending caused by pure couples unaccompanied by shearing force. Since, in general, the deflection due to the shearing forces is extremely small compared to that due to the flexural forces, it will be neglected in this discussion, and all beams will be treated as cases of simple flexure.

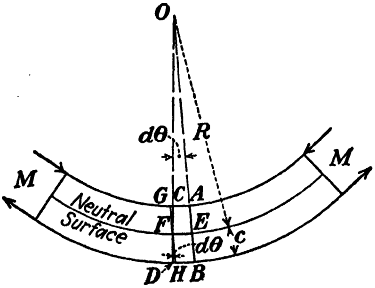


FIG. 102.

The following assumptions are made in the case of simple flexure:

1. Plane transverse sections remain plane after bending.
2. The modulus of elasticity in tension is the same as that in compression.
3. The proportional elastic limit is not exceeded.

Figure 102 represents a beam which is subjected to simple bending due to the couples M, M. AB and CD are two transverse sections, originally parallel, and EF is the original distance between them. Lines AB and CD when produced intersect at point O, and the angle between them is $d\theta$. Line GFH is drawn through point F parallel to line AB. Angle DFH is equal to angle $d\theta$, since their sides are parallel. The unit elongation in length BD is

$$\delta = \frac{DH}{FE} = \frac{c\,d\theta}{R\,d\theta} = \frac{c}{R}.$$

Also, since

$$\delta = \frac{S}{E} \quad \text{and} \quad S = \frac{Mc}{I},$$

$$\delta = \frac{Mc}{EI} = \frac{c}{R}.$$

$$\frac{1}{R} = \frac{M}{EI}.$$

The equation just derived gives the relation between the radius of curvature of the beam at any section and the moment at that section. The modulus of elasticity E is, of course, assumed to be constant. Then if the moment of inertia I is also constant, the radius of curvature R varies inversely with the moment M. If the moment M is zero, the radius of curvature R becomes infinity and the elastic curve is a straight line. As the moment M increases, the radius of curvature R decreases, and its smallest value occurs where the moment is a maximum.

If the moment M is constant throughout the length of a beam, the radius of curvature R is constant and the line FE of the neutral surface is the arc of a circle. If the moment M varies at different sections in the beam, the radius of curvature R varies inversely.

PROBLEMS

1. A wooden cantilever beam is 12 in. wide, 2 in. deep, and 10 ft. long, and carries a total uniformly distributed load of 20 lb. per linear ft. If $E = 1{,}800{,}000$ lb./in.^{2}, what is the radius of curvature of the beam at the fixed end? What is the radius of curvature 1 ft. from the free end?

Ans. 100 ft.; 10,000 ft.

2. A band saw $\frac{1}{16}$ in. thick runs on pulleys 2.5 ft. in diameter. $E = 30{,}000{,}000$ lb./in.^{2}. Compute the stress caused by bending around the pulleys. *Ans.* 62,370 lb./in.^{2}.

3. If the elastic limit of steel wire is 50,000 lb./in.^{2}, and $E = 29{,}000{,}000$ lb./in.^{2}, compute the diameter of the smallest circle into which a No. 9 wire (0.148 in. diameter) may be bent without receiving a permanent set.

Ans. 7.16 ft.

58. Curvature, Slope, and Deflection.—It is shown in works on differential calculus that the curvature of any curve, in terms of the rectangular coordinates x and y is

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

For beams as usually loaded the quantity dy/dx is very small compared with unity, so higher powers than the first may be neglected without appreciable error. The expression given above will then reduce to

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{(approx.).}$$

It was shown in Art. 57 that $1/R = M/EI$, so

$$EI\frac{d^2y}{dx^2} = M.$$

In the equation just derived, y is the ordinate of any point on the elastic curve, considered as positive upward, and x is the abscissa of the same point, considered as positive to the right. The quantity dy/dx gives the slope of the elastic curve at any point, and its ordinates may be plotted to give the slope curve. The quantity d^2y/dx^2 gives the slope of the slope curve. If the beam is convex upward, it is evident that the center of curvature is below the beam, the value of dy/dx, the slope of the elastic curve, decreases from left to right, and consequently the sign of d^2y/dx^2, the slope of the slope curve, is essentially negative. If the beam is concave upward, the center of curvature is above the beam, the value of dy/dx increases from left to right, and consequently the sign of the second derivative d^2y/dx^2 is essentially positive.

Horizontal beams which are convex upward, therefore, have negative moment and those which are concave upward have positive moment. If the moment at a section is determined by the moment of the forces on the left of the section, clockwise moment is positive and counterclockwise moment is negative.

In solving this equation, the moment M is expressed in terms of x and the resulting equation is integrated twice, by which the value of y is determined in terms of x. Since in the first integration the limits of the slope dy/dx are unknown, and in the second integration the limits of y are unknown, constants of integration must be employed. The values of these constants of integration are determined by substituting corresponding known values of the variables.

As already stated, the modulus of elasticity E is considered to be constant, and, except where otherwise stated, the beam is considered to be uniform in cross section, so the moment of inertia I is constant.

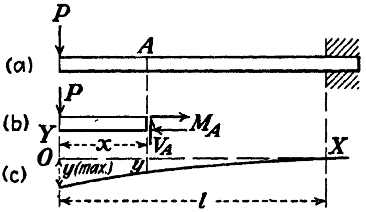


FIG. 103.

59. Cantilever Beam with Concentrated Load at the End.— Figure 103(a) represents a cantilever beam fixed at the right end and loaded with a concentrated load P at the left end. The origin of coordinates is taken at O, Fig. 103(c), the position of the free end of the beam before the load was applied. The moment at any point A, distant x from the left end of the beam, is $M_A = -Px$ as shown in the free-body diagram, Fig. 103(b). The differential equation becomes

$$EI\frac{d^2y}{dx^2} = -Px.$$

By integration,

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1.$$

Since the slope $dy/dx = 0$ at the fixed end where $x = l$,

$$0 = -\frac{Pl^2}{2} + C_1.$$

$$C_1 = \frac{Pl^2}{2}.$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{Pl^2}{2}.$$

By a second integration,

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} + C_2.$$

Since $y = 0$ at the fixed end where $x = l$,

$$0 = -\frac{Pl^3}{6} + \frac{Pl^3}{2} + C_2.$$

$$C_2 = -\frac{Pl^3}{3}.$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} - \frac{Pl^3}{3}.$$

This expression is called the *equation of the elastic curve* of the beam, and gives the deflection y of any point on the beam at a distance x from the free end.

The maximum value of y is at the left end of the beam where $x = 0$. With this value of x,

$$\text{Max. } y = -\frac{Pl^3}{3EI}.$$

Since E is in units of pounds per square inch, l must be used in inches and P in pounds.

PROBLEMS

1. Derive the expression for the slope of the beam at the free end and at the middle. *Ans.* $\frac{dy}{dx} = \frac{Pl^2}{2EI}; \frac{dy}{dx} = \frac{3Pl^2}{8EI}$.

2. Derive the expression for the deflection at the end of the beam in terms of the maximum stress S. *Ans.* $y = \frac{Sl^2}{3Ec}$.

3. Use the origin of coordinates at the free end of the beam after it is loaded and derive the equation of the elastic curve.

Ans. $EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2}$.

4. A wooden cantilever beam 4 in. wide, 6 in. deep, and 8 ft. long carries a load of 300 lb. at the free end. $E = 1{,}200{,}000$ lb./in.^{2}. Compute the slope at the free end, the deflection at the free end, and the radius of curvature at the fixed end. *Ans.* 0.016; 1.024 in.; 250 ft.

5. A piece of steel water pipe 3 ft. long is 1.66 in. outside diameter and 1.38 in. inside diameter. If used as a cantilever beam, what load at the free end will deflect it ½ in.? $E = 29{,}000{,}000$ lb./in.^{2} *Ans.* 182 lb.

6. A 10-in. 40-lb. I-beam 10 ft. long is used as a cantilever beam with web vertical. If $E = 29{,}000{,}000$ lb./in.^{2} and the allowable stress is 18,000 lb./in.^{2}, what is the maximum permissible deflection by a vertical load at the end? *Ans.* 0.595 in.

7. Compute the load necessary to cause the deflection given for the beam in Problem 6. What horizontal load at the free end will produce the same amount of deflection? *Ans.* 4740 lb.; 282 lb.

60. Cantilever Beam with Uniformly Distributed Load.—Figure 104(a) represents a cantilever beam fixed at the right end and loaded with a uniformly distributed load of w per unit length. The origin of coordinates is taken at O, Fig. 104(c), the position of the free end of the beam before the load was applied. The moment at any point A, distant x from the left end of the beam, is $M_A = -wx^2/2$, as shown in the free body diagram, Fig. 104(b).

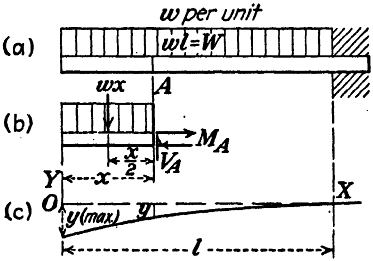


FIG. 104.

The differential equation becomes

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}.$$

By integration,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1.$$

Since the slope $dy/dx = 0$ at the fixed end where $x = l$,

$$0 = -\frac{wl^3}{6} + C_1.$$

$$C_1 = \frac{wl^3}{6}.$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}.$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} + C_2.$$

Since $y = 0$ at the fixed end where $x = l$,

$$0 = -\frac{wl^4}{24} + \frac{wl^4}{6} + C_2.$$

$$C_2 = -\frac{wl^4}{8}.$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}.$$

This expression is called the *equation of the elastic curve* of the beam, and gives the deflection y at any point at a distance x from the free end.

The maximum value of y is at the left end of the beam where $x = 0$. With this value of x,

$$\text{Max. } y = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI},$$

since the total load is $wl = W$.

Since E is in units of pounds per square inch, l must be used in inches and w in pounds per linear inch.

PROBLEMS

1. Derive the expression for the slope of the beam at the free end and at the middle. *Ans.* $\frac{dy}{dx} = \frac{Wl^2}{6EI}$; $\frac{dy}{dx} = \frac{7Wl^2}{48EI}$.

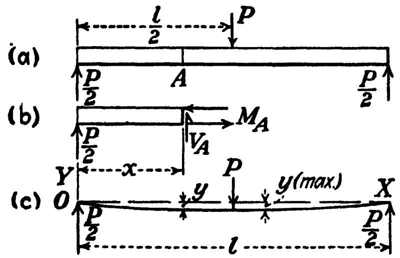


FIG. 105.

2. Derive the expression for the maximum deflection in terms of the maximum stress S. *Ans.* $y = \frac{Sl^2}{4Ec}$.

3. Use the origin of coordinates at the free end of the beam after it is loaded and derive the equation of the elastic curve.

Ans. $EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6}$.

4. A wooden cantilever beam is 10 in. wide, 2 in. deep, and 12 ft. long. If $E = 1{,}500{,}000$ lb./in.^{2}, what uniformly distributed load per linear foot will cause a deflection of 5 in. at the free end? *Ans.* 11.13 lb./ft.

5. If $E = 30{,}000{,}000$ lb./in.^{2}, what must be the diameter of a circular steel cantilever beam 5 ft. long if the deflection due to its own weight is not to exceed 0.001 in.? *Ans.* 15.65 in.

6. A 12-in. 31.8-lb. I-beam 10 ft. long, used as a cantilever beam, supports a superimposed load of 1000 lb. per linear foot. Using E as 29,000,000 lb./in.^{2}, compute the deflection of the free end due both to its own weight and to the superimposed load. *Ans.* 0.357 in.

7. With the same load per linear foot as that given in Problem 6, how long can the beam be made if the maximum deflection is not to exceed 1 in.? Is the beam safe? Is the computation valid?

Ans. 12.94 ft.; $S = 29{,}000$ lb./in.^{2}.

61. Simple Beam with Concentrated Load at the Middle.—Figure 105(*a*) represents a simple beam of length l, with a concentrated load P at the middle. The origin is taken at O, Fig. 105(*c*), the left end of the beam. Each reaction is $P/2$ and the moment at A is $M_A = Px/2$, as shown in the free-body diagram, Fig. 105(*b*). This expression for the moment is valid only from the left end of the beam to the middle, that is, for values of x between zero and $l/2$.

$$EI\frac{d^2y}{dx^2} = +\frac{Px}{2}.$$

By integration,

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1.$$

By symmetry, $dy/dx = 0$ at the middle where $x = l/2$.

$$C_1 = -\frac{Pl^2}{16}.$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16}.$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} + C_2.$$

Since $y = 0$ where $x = 0$, $C_2 = 0$.

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}.$$

This expression is called the *equation of the elastic curve* of the left half of the beam, and gives the deflection y for any given value of x between zero and $l/2$. The maximum value of y is at the middle where $x = l/2$.

$$\text{Max. } y = -\frac{Pl^3}{48EI}.$$

By symmetry, any point on the right half of the beam is deflected the same amount as the corresponding point on the left half of the beam.

If the beam overhangs one of the supports an amount l_1 and the weight of the beam is neglected, the overhanging end will be raised a distance $y_1 = l_1 \tan \theta$ above the level of the supports, θ being the angle of slope of the beam at the support.

$$y_1 = l_1 \tan \theta = \frac{Pl_1l^2}{16EI}.$$

PROBLEMS

1. A wooden beam 4 in. wide, 6 in. deep, and 15 ft. long is supported at the ends and carries a load of 700 lb. at the middle. Compute the deflection at the middle and at each third point due to the 700-lb. load if $E = 1{,}100{,}000$ lb./in.^{2}. *Ans.* 1.074 in.; 0.915 in.

2. Derive the expression for the maximum deflection in terms of the maximum stress S. *Ans.* $y = \dfrac{Sl^2}{12Ec}$.

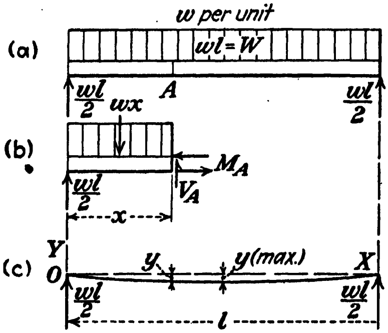


FIG. 106.

3. A wooden test beam 1.75 in. wide, 1.71 in. deep, and 28 in. long between supports, deflected 0.068 in. with a load of 200 lb. at the middle. Compute E. *Ans.* 1,846,000 lb./in.^{2}.

4. A solid steel line shaft is 3 in. in diameter and 10 ft. between hangers. If $E = 30{,}000{,}000$ lb./in.^{2}, how much will a pulley weighing 30 lb. deflect the shaft if placed midway between the supports? *Ans.* 0.00905 in.

5. A 20-in. 100-lb. I-beam 25 ft. long is supported at the ends and carries a load of 40,000 lb. at the middle. Compute the deflection at the middle due to the 40,000-lb. load if $E = 29{,}000{,}000$ lb./in.^{2}. What side thrust at the middle would cause the same amount of deflection laterally?
Ans. 0.472 in.; 1272 lb.

6. A 5-in. 10-lb. I-beam 10 ft. long is supported at the ends. If $E = 29{,}000{,}000$ lb./in.^{2}, what is the maximum deflection if it is loaded with a concentrated load at the middle to cause a stress of 18,000 lb./in.^{2}?
Ans. 0.298 in.

7. A wooden plank 12 in. wide, 2 in. deep, and 16 ft. long is supported at the left end and at a point 4 ft. from the right end. If it is loaded with a load midway between the supports to cause a maximum stress of 1100 lb./in.^{2}, how much is the right end deflected above the level of the supports? $E = 1{,}600{,}000$ lb./in.^{2}. *Ans.* 1.188 in.

62. Simple Beam with Uniformly Distributed Load.—Figure 106(a) represents a simple beam of span l, with a uniformly distributed load of w per unit length. The total weight is wl or W, and each reaction is $wl/2$ by symmetry. The origin of coordinates is taken at O, Fig. 106(c), the left end of the beam. The moment at A of all the forces on the beam on the left of A is $M_A = wlx/2 - wx^2/2$, as shown in the free-body diagram, Fig. 106(b).

The differential equation becomes

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}.$$

By integration,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1.$$

By symmetry, $dy/dx = 0$ where $x = l/2$.

$$C_1 = -\frac{wl^3}{24}.$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}.$$

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2.$$

Since $y = 0$ where $x = 0$, $C_2 = 0$.

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}.$$

This expression is called the *equation of the elastic curve* of the beam, and gives the deflection y for any given value of x. The maximum value of y is at the middle where $x = l/2$.

$$\text{Max. } y = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI}.$$

If the beam overhangs one of the supports a distance l_1 and this length l_1 carries no load, its own weight being considered negligible, the overhanging part remains straight and the outer end is raised an amount of $y_1 = l_1 \tan \theta$ above the level of the supports, θ being the angle of slope at the supports.

$$y_1 = l_1 \tan \theta = \frac{Wl_1l^2}{24EI}.$$

PROBLEMS

1. Derive the expression for the deflection of the beam at the quarter points. *Ans.* $y = -\frac{57Wl^3}{6144EI}$.

2. Derive the expression for the maximum deflection y in terms of the maximum stress S. *Ans.* $y = \frac{5Sl^2}{48Ec}$.

3. A wooden floor joist 3 in. wide, 16 in. deep, and 20 ft. long is supported at the ends and loaded with a load of 200 lb. per linear ft. $E = 1{,}400{,}000$ lb./in.^{2}. Compute the maximum deflection. *Ans.* 0.502 in.

4. An A. S. C. E. railroad rail weighing 90 lb. per yard is 39 ft. long and rests on supports at each end. $E = 29{,}000{,}000$ lb./in.^{2} and $I = 34.39$ in.^{4}. Compute the deflection at the middle. *Ans.* 1.57 in.

5. If $E = 30{,}000{,}000$ lb./in.^{2}, how much will a hollow steel shaft 2 in. outside diameter and 1.5 in. inside diameter be deflected by its own weight in a span of 8 ft.? *Ans.* 0.0268 in.

6. A simple beam 20 ft. long consists of a pair of 8-in. 11.5-lb. channels placed back to back with the webs vertical. The beam is loaded with a uniformly distributed load to cause a maximum stress of 16,000 lb./in.^{2}. Compute the maximum deflection if $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 0.828 in.

7. A floor is supported by 10-in. 40-lb. I-beams 30 ft. long and spaced 16 in. apart. $E = 29{,}000{,}000$ lb./in.^{2}. Compute the load per square foot of floor area which will cause a maximum deflection of 0.5 in. *Ans.* 94.3 lb.

8. A 15-in. 42.9-lb. I-beam 30 ft. long is supported at the right end and at a point 5 ft. from the left end. It is loaded between the supports with a total uniformly distributed load of 25,000 lb. Using E as 29,000,000 lb./in.^{2}, compute the amount the overhanging end is raised above the level of the supports. Neglect the weight of the overhanging part of the beam. *Ans.* 0.439 in.

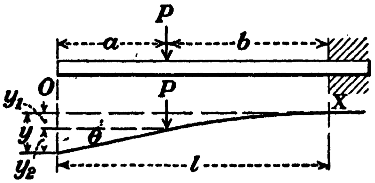


FIG. 107.

63. Cantilever Beam with Concentrated Load at Any Point.—Figure 107 represents a cantilever beam of length l with a load P at a point distant a from the free end and b from the fixed end. The deflection y at the end is made up of two parts, y_1 and y_2. By Art. 59, the deflection

$$y_1 = -\frac{Pb^3}{3EI}.$$

If the weight of the beam is negligible, the length a of the beam is straight and has the same slope as the beam under the load. By Art. 59 this is

$$\frac{dy}{dx} = \frac{Pb^2}{2EI}.$$

The deflection y_2 is equal to $-a \tan \theta$, or $-a\dfrac{dy}{dx}$.

$$y_2 = -\frac{Pab^2}{2EI}.$$

The total deflection at the end is given by the expression,

$$y = y_1 + y_2 = -\frac{Pb^2}{EI}\left(\frac{a}{2} + \frac{b}{3}\right).$$

PROBLEMS

1. A cantilever beam of length l has a load P at the middle. Compute the deflection at the end and the deflection under the load.

Ans. $y = -\dfrac{5Pl^3}{48EI}$; $y_1 = -\dfrac{Pl^3}{24EI}$.

2. A wooden cantilever beam 6 in. wide, 2 in. deep, and 8 ft. long carries a load of 80 lb. at a point 3 ft. from the free end. $E = 1{,}600{,}000$ lb./in.^{2}. Compute the deflection of the free end. *Ans.* 1.71 in.

3. A 6-in. 12.5-lb. I-beam 10 ft. long is used as a cantilever beam. If $E = 29{,}000{,}000$ lb./in.^{2}, how far from the fixed end may a load of 1000 lb. be placed to cause a deflection of 0.5 in. at the free end? *Ans.* 82.7 in.

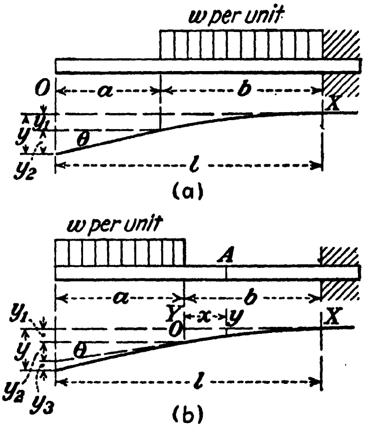


FIG. 108.

64. Cantilever Beam with a Partial Uniformly Distributed Load.—Figure 108(a) represents a cantilever beam of length l loaded with a uniformly distributed load of w per unit length over a distance b from the fixed end, the remaining portion a being without load. By Art. 60, the deflection

$$y_1 = -\frac{Wb^3}{8EI},$$

and the deflection

$$y_2 = -\frac{Wab^2}{6EI}.$$

$$y = y_1 + y_2 = -\frac{Wb^2}{2EI}\left(\frac{a}{3} + \frac{b}{4}\right).$$

Figure 108(*b*) represents a cantilever beam of length l loaded with a uniformly distributed load of w per unit length over a distance a from the free end, the remaining portion b being without load. Let the origin be taken at O, the position of the beam at the right end of the load before the load was applied. The moment at point A, distant x from O, is

$$M_A = -\frac{wa^2}{2} - wax.$$

$$EI\frac{d^2y}{dx^2} = -\frac{wa^2}{2} - wax.$$

$$EI\frac{dy}{dx} = -\frac{wa^2x}{2} - \frac{wax^2}{2} + C_1.$$

The slope $dy/dx = 0$ where $x = b$.

$$C_1 = \frac{wabl}{2}.$$

$$EI\frac{dy}{dx} = -\frac{wa^2x}{2} - \frac{wax^2}{2} + \frac{wabl}{2}.$$

$$EIy = -\frac{wa^2x^2}{4} - \frac{wax^3}{6} + \frac{wablx}{2} + C_2.$$

The deflection $y = 0$ where $x = b$.

$$EIy = -\frac{wa^2x^2}{4} - \frac{wax^3}{6} + \frac{wablx}{2} + \frac{wa^2b^2}{4} + \frac{wab^3}{6} - \frac{wab^2l}{2}.$$

The deflection $y_1 = -\frac{W}{EI}\left(\frac{b^2l}{2} - \frac{ab^2}{4} - \frac{b^3}{6}\right).$

The deflection $y_2 = -a \tan \theta = -\frac{W}{EI}\left(\frac{abl}{2}\right).$

The deflection $y_3 = -\frac{W}{EI}\left(\frac{a^3}{8}\right).$

Since $l = a + b$, the total deflection is

$$y = -\frac{W}{EI}\left(\frac{a^3}{8} + \frac{a^2b}{2} + \frac{3ab^2}{4} + \frac{b^3}{3}\right).$$

PROBLEMS

1. A wooden cantilever beam 6 in. wide, 10 in. deep, and 12 ft. long is loaded with a uniformly distributed load of 240 lb. per linear ft. extending over a length of 8 ft. from the fixed end. $E = 1{,}250{,}000$ lb./in.². Compute the deflection at the end due to this load. *Ans.* 0.565 in.

2. A 7-in. 20-lb. I-beam 10 ft. long is used as a cantilever beam. How much load per foot may be placed on the beam, extending 4 ft. from the free end, to cause a deflection of ½ in. at the free end if E is 29,000,000 lb./in.²? *Ans.* 372.5 lb./ft.

65. Simple Beam with Concentrated Load at Any Point.—Figure 109 represents a simple beam loaded with a concentrated load P at a point on the right of the middle of the beam. If the distance from the load to the left end is a and that to the right end is b, the left reaction is Pb/l and the right reaction is Pa/l.

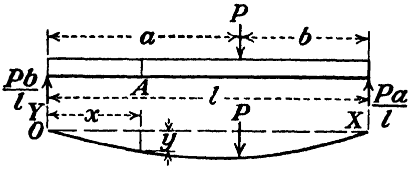


FIG. 109.

If the origin of coordinates is taken at the left end of the beam, the moment at any point on the left portion of the beam is

$$M_a = \frac{Pbx}{l},$$

and that at any point on the right portion is

$$M_b = \frac{Pbx}{l} - P(x - a).$$

For the left portion of the beam:	For the right portion of the beam:
$\frac{EI}{P}\frac{d^2y}{dx^2} = \frac{bx}{l}.$	$\frac{EI}{P}\frac{d^2y}{dx^2} = \frac{bx}{l} - (x - a).$
$\frac{EI}{P}\frac{dy}{dx} = \frac{bx^2}{2l} + C_1.$	$\frac{EI}{P}\frac{dy}{dx} = \frac{bx^2}{2l} - \frac{(x-a)^2}{2} + C_2.$

The slope dy/dx under the load is the same for both portions.

$$\frac{ba^2}{2l} + C_1 = \frac{ba^2}{2l} - \frac{(a-a)^2}{2} + C_2.$$

$$C_1 = C_2.$$

$\frac{EI}{P}y = \frac{bx^3}{6l} + C_1x + C_3.$	$\frac{EI}{P}y = \frac{bx^3}{6l} - \frac{(x-a)^3}{6} + C_1x + C_4.$

At the left end, $y = 0$, $x = 0$, and therefore $C_3 = 0$.

The deflection under the load is the same for both portions.

$$\frac{ba^3}{6l} + C_1 a = \frac{ba^3}{6l} - \frac{(a-a)^3}{6} + C_1 a + C_4.$$
$$C_4 = 0.$$

For the right portion of the beam, $y = 0$ where $x = l$.

$$0 = \frac{bl^2}{6} - \frac{b^3}{6} + C_1 l.$$
$$C_1 = \frac{b^3}{6l} - \frac{bl}{6}.$$

With this value of C_1, the equation of the elastic curve for the left portion of the beam becomes

$$\frac{EI}{P}y = \frac{bx^3}{6l} - \frac{blx}{6} + \frac{b^3x}{6l}.$$

If $x = a$, the deflection under the load is

$$y_P = \frac{Pa^2b^2}{3EIl}.$$

The slope equation for the left portion of the beam becomes

$$\frac{EI}{P}\frac{dy}{dx} = \frac{bx^2}{2l} + \frac{b^3}{6l} - \frac{bl}{6}.$$

Since the slope equals zero at the point which has a maximum deflection,

$$0 = \frac{bx^2}{2l} + \frac{b^3}{6l} - \frac{bl}{6}.$$
$$x = \sqrt{\frac{l^2 - b^2}{3}}.$$

It will be seen that the extreme limits of the value of the distance x from the left end to the point of maximum deflection as b varies from $l/2$ to zero are $0.5l$ and $0.577l$. By using the value of x just obtained in the equation of the elastic curve, the value of the maximum deflection is given by the expression,

$$\text{Max. } y = -\frac{Pb}{27EIl}(l^2 - b^2)\sqrt{3(l^2 - b^2)}.$$

The deflection at the middle is given by the expression,

$$y_c = -\frac{Pb}{EI}\left(\frac{l^2}{16} - \frac{b^2}{12}\right).$$

The error in using the simpler expression for y_c, the deflection at the middle, instead of the more complicated expression for the maximum deflection y, is practically negligible.

PROBLEMS

1. A simple wooden beam 4 in. wide, 18 in. deep, and 20 ft. long carries a load of 4000 lb. at a point 4 ft. from the right end. Locate the point of maximum deflection. Using the value of E as 1,600,000 lb./in.², compute the maximum deflection and the deflection at the middle.

Ans. 11 ft. 4 in.; 0.215 in.; 0.210 in.

2. How near the middle of the beam could the load be placed on the beam described in Problem 1 if the allowable deflection at the middle is 0.3 in.?

Ans. 45.7 in.

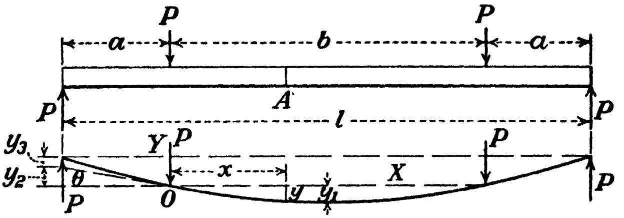


FIG. 110.

66. Beam with Constant Moment.

66. Beam with Constant Moment.—Figure 110 represents a beam of length l supported at the ends, with two equal loads, P, P, each at a distance a from the end and at a distance b apart. Let the origin be taken at O, the position of the left load after the loads are applied. By symmetry, each reaction is P, so the moment at any point A between the loads is

$$M_A = Pa.$$

$$EI\frac{d^2y}{dx^2} = Pa.$$

$$EI\frac{dy}{dx} = Pax + C_1.$$

At the middle of the beam, by symmetry, $dy/dx = 0$ and $x = b/2$.

$$C_1 = -\frac{Pab}{2}.$$

$$EI\frac{dy}{dx} = Pax - \frac{Pab}{2}.$$

$$EIy = \frac{Pax^2}{2} - \frac{Pabx}{2} + C_2.$$

At the origin O, $y = 0$ and $x = 0$, so $C_2 = 0$.

$$EIy = \frac{Pax^2}{2} - \frac{Pabx}{2}.$$

The maximum deflection is at the middle where $x = b/2$, and its distance below the level of the loads is given by the expression,

$$y_1 = -\frac{Pab^2}{8EI}.$$

The deflection of the loads below the ends of the beam is made up of two parts, y_2 and y_3.

$$y_2 = -a \tan \theta = -\frac{Pa^2b}{2EI}.$$

The deflection y_3 is the deflection of a cantilever beam with a load P at the end from the tangent at O.

$$y_3 = -\frac{Pa^3}{3EI}.$$

The total deflection of the middle point of the beam below the level of the ends is given by the expression,

$$\text{Max. } y = y_1 + y_2 + y_3 = -\frac{Pa}{EI}\left(\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{8}\right).$$

PROBLEMS

1. A beam of length l is supported at the ends. Compute the deflection at the middle when two loads, each of amount P, are placed at the quarter points. *Ans.* $y = -\frac{11Pl^3}{384EI}$.

2. Compute the deflection at the middle of the beam described in Problem 1 if the loads are placed at the third points. *Ans.* $y = -\frac{23Pl^3}{648EI}$.

3. A wooden beam 8 in. wide, 20 in. deep, and 24 ft. long, supported at the ends, has a load of 10,000 lb. placed at each quarter point. Using E as 1,200,000 lb./in.^{2} and neglecting the weight of the beam, compute the deflection of the loads below the level of the ends and the deflection of the middle below the level of the ends. *Ans.* 0.778 in.; 1.069 in.

4. A 12-in. 31.8-lb. I-beam 40 ft. long is supported at points 5 ft. from each end, and carries a load of 6000 lb. at each end. Compute the upward deflection of the middle above the level of the supports due to the loads. *Ans.* 0.931 in.

67. Comparative Stiffness of Beams.—The stiffness of a beam is the inverse of the deflection. It will be noticed that the deflection varies inversely as E and I, so the stiffness varies directly as E and I. It will also be noticed that the deflection varies directly as the cube of the length of the beam, so the stiffness varies inversely as the cube of the length.

With the same amount of load, whether concentrated or uniformly distributed, acting upon beams of the same length, shape, and material, the relative deflections and relative stiffnesses for the four principal cases vary as shown in the following table:

TABLE SHOWING RELATIVE DEFLECTIONS AND STIFFNESSES OF BEAMS

Kind of beam	Loading	Deflection	Relative deflection	Relative stiffness
Cantilever.....	P at end	$\frac{Pl^3}{3EI}$	25.6	1
Cantilever.....	W distributed	$\frac{Wl^3}{8EI}$	9.6	2.67
Simple........	P at middle	$\frac{Pl^3}{48EI}$	1.6	16
Simple........	W distributed	$\frac{5Wl^3}{384EI}$	1	25.6

PROBLEMS

1. Compare the stiffness of a beam 8 in. wide and 2 in. deep with that of one 4 in. wide and 4 in. deep, and with that of one 2 in. wide and 8 in. deep, all of the same length and material. *Ans.* 1:4:16.

2. Show that a square steel beam placed with the sides vertical and horizontal is just as stiff as it is when placed with the diagonals vertical and horizontal. How do the strengths compare? *Ans.* 1 to 0.707.

3. If the 4- by 4-in. beam in Problem 1 is 10 ft. long, how long must each of the others be made so that they will have the same deflection when supported and loaded in the same way. *Ans.* 6.3 ft.; 15.87 ft.

68. Shear Center for Channel Section.—If the cross section of a beam has only one axis of symmetry and if the load is applied in a plane normal to this axis and through the centroid of the section, shearing forces are induced which cause the beam to rotate about its linear axis. The rolled channel section, the H-section with unequal flanges, and the angle section with equal legs are familiar examples of such sections. If this rotation about the linear axis is to be avoided, it is necessary for the load to be applied through a point in the cross section at some distance

e from the centroid. This point of application of the load, in order that there shall be no twist of the beam as it deflects, is called the *shear center.*

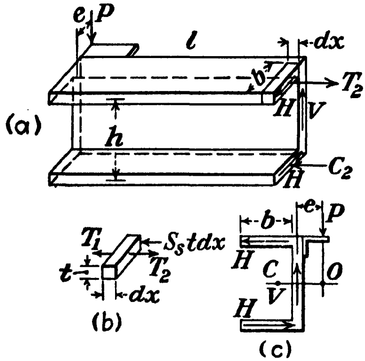


FIG. 111.

In Fig. 111(a) is shown a channel section used as a cantilever beam and separated from its fixed support to make it a free body. The beam is free at the left end and is loaded with a vertical load P. Where the fixed support is removed at the right end, the forces which the part removed must exert upon the free body are shown acting at the cut section. The vertical force which must be applied is the upward vertical shear V. As has already been shown, this is practically all in the web and is here assumed to be acting at the middle. If the weight of the beam is neglected, $V = P$. There is also developed a tensile force in the upper part of the section and a compressive force in the lower part of the section, the moment of which is equal to Pl. In the figure, T_2 is only the part of this tensile force which is acting in the upper flange. There are also tensile forces in the upper part of the web which vary from a maximum value at the outer fiber to zero at the neutral axis, and the corresponding compressive forces in the lower part of the web. In addition to these forces, there are developed horizontal lateral shearing forces, H, H, as will now be shown. Let a part of the upper flange dx in length at the fixed end be taken as the free body, as shown in Fig. 111(b). Only the linear horizontal forces are shown. Force T_2 is the total horizontal tension and is approximately equal to $btM_2h/2I$, M_2 being the moment at the section. Force T_1 is slightly less, and is approximately equal to $btM_1h/2I$. In order to produce equilibrium horizontally, there is developed a longitudinal shearing force S_s in unit intensity. Since the sum of the longitudinal forces equals zero,

$$S_s t\,dx = T_2 - T_1 = (M_2 - M_1)\frac{bth}{2I}.$$

Since $M_2 - M_1 = dM$, and $dM/dx = V$,

$$S_s = \frac{Vbh}{2I}.$$

This longitudinal shearing stress develops an equal horizontal lateral shearing stress, acting toward the left in the upper flange. Since the shearing stress reduces to zero at the free edge of the flange, the total lateral shearing force $H = S_s bt/2 = Vb^2ht/4I$. In the lower flange there is developed an equal lateral force H in the opposite direction, and the moment of this couple about any longitudinal axis is $Hh = Vb^2h^2t/4I$.

In order that the moment of force P may balance this developed couple, it is necessary for P to be applied far enough out from the web on the side opposite to the flange, so that the moment Pe of the couple PV about a longitudinal axis in the middle of the web is equal and opposite to the moment Hh.

Since $P = V$,

$$e = \frac{b^2h^2t}{4I}.$$

Point O through which force P must act is the *shear center* of the section.

EXAMPLE

A channel section is 10 in. deep and has flanges 6 in. wide and ½ in. thick and a web ¾ in. thick. Compute the distance e from the center of the web to the shear center O.

Solution:

$$I = \tfrac{1}{12}\, 6 \times 10^3 - \tfrac{1}{12}\, 5.25 \times 9^3 = 181.$$

$b = 5.25$ in., $h = 9.5$ in., $t = 0.5$ in.

$$e = \frac{b^2h^2t}{4I} = \frac{5.25^2 \times 9.5^2 \times 0.5}{4 \times 181} = 1.72 \text{ in.}$$

The distance from the edge of the web is $1.72 - 0.375 = 1.345$ in. Since $\bar{y}$ from the outer edge of the web to the centroid of the section is 1.61 in., the distance of the shear center from the centroid is $1.345 + 1.61 = 2.955$ in.

PROBLEMS

1. Compute the distance e for a 6-in. 15-lb. shipbuilding channel. The web is 0.35 in. thick and the flanges are 3.5 in. wide and average 0.375 in. thick. *Ans.* 1.18 in.

2. A 12-in. 50-lb. shipbuilding channel has a web 0.84 in. thick and flanges 4.14 in. wide and 0.7 in. thick. If a load of 5000 lb. is applied in a plane through the centroidal axis of the section, what twisting moment is caused? *Ans.* 7735 in.-lb.

69. Shear Center for H-Section with Unequal Flanges.—If a beam has an H-section with the flanges unequal, and if the axis of symmetry is to be the neutral axis, as shown in Fig. 112, the shear center is between the centroid of the section and the center of the flange having the larger moment of inertia. If the web is thin compared to the vertical dimensions of the flanges, the amount of the vertical shear carried by it is very small and can be neglected without appreciable error.

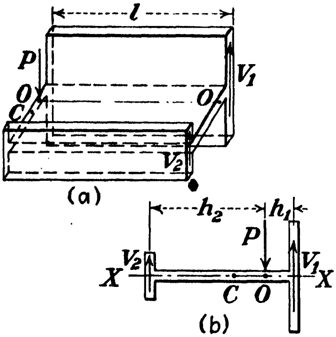


FIG. 112.

With the beam as the free body and neglecting its weight,

$$\Sigma F_y = 0 \text{ gives}$$
$$V_1 + V_2 = P.$$

If the axis of symmetry, X-X, is to deflect vertically without twisting, it is necessary that the moment with respect to O must be zero.

$$V_1 h_1 = V_2 h_2, \qquad \text{or} \qquad \frac{V_1}{V_2} = \frac{h_2}{h_1}.$$

Also, since the radii of curvature, R_1 and R_2, of the two flanges must be equal, and since $R = EI/M$,

$$\frac{EI_1}{M_1} = \frac{EI_2}{M_2}.$$

Since $M_1 = V_1 l$ and $M_2 = V_2 l$,

$$\frac{I_1}{V_1} = \frac{I_2}{V_2}, \qquad \text{or} \qquad \frac{V_1}{V_2} = \frac{I_1}{I_2}.$$

From this and the preceding equation for V_1/V_2,

$$\frac{h_2}{h_1} = \frac{I_1}{I_2}.$$

If the two flanges have equal values of I even though their areas are different, the shear center is midway between their centroids. If the area of one flange becomes zero so that the cross section is a T with the stem horizontal, the shear center is at the middle point of the remaining flange.

EXAMPLE

In Fig. 112 let the right flange be 0.6 in. wide and 10 in. deep, the web 8 in. wide and 0.5 in. deep, and the left flange 2 in. wide and 4 in. deep. Locate the centroid and the shear center from the outer edge of the right flange.

Solution:

$$A_1 = 6 \text{ sq. in.}, \quad A_2 = 4 \text{ sq. in.}, \text{ and } A_3 = 8 \text{ sq. in.}$$

$$18\bar{x} = 6 \times 0.3 + 4 \times 4.6 + 8 \times 9.6.$$

$$\bar{x} = 5.39 \text{ in.}$$

$$I_1 = \tfrac{1}{12}0.6 \times 1000 = 50. \qquad I_2 = \tfrac{1}{12}2 \times 64 = 10.67.$$

$$h_2 = \frac{50}{10.67}h_1 = 4.69h_1.$$

$$5.69h_1 = 9.3.$$

$$h_1 = 1.637 \text{ in.}$$

From the outer edge of the right flange,

$$x_0 = 1.937 \text{ in.}$$

PROBLEMS

1. Solve the example above if both flanges are ½ in. thick, all other dimensions remaining the same. *Ans.* $\bar{x} = 3.341$ in.; $x_0 = 0.761$ in.

2. In the H-beam of Problem 1, how deep must the left flange be in order that the shear center shall be 1 in. from the inner edge of the right flange, all other dimensions remaining unchanged. *Ans.* 5.565 in.

70. Shear Center for Angle Section.—The beam shown in Fig. 113 consists of an angle section with equal legs, fixed at the right end and loaded with load P at the left end. The axis of symmetry is horizontal. In the figure, the beam is shown as a free body, with the shearing forces P_1, P_1 at the cut section indicated. (The forces normal to the cut section do not enter into the equations used and hence are not shown.) By the expression for horizontal shear, the unit shearing stress at point B in the cut section is

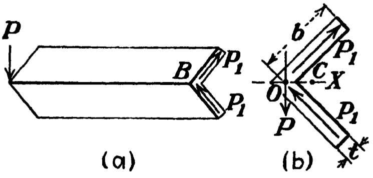


FIG. 113.

$$S_s = \frac{P}{I_x t}\bar{y}A' = \frac{Pb^2}{I_x 2\sqrt{2}}.$$

Since the unit shearing stress S_s varies as the ordinates to a parabola, as shown in Art. 54, the total shearing force P_1 in the leg is

$$P_1 = \frac{2}{3}S_s bt = \frac{Pb^3t}{I_x 3\sqrt{2}}.$$

The shear center is at the intersection of P_1 in the upper leg and P_1 in the lower leg, and is therefore at point O. Also, since P is equal to the resultant of the two shearing forces P_1,

$$P = \frac{Pb^3t}{3I_x}.$$

$$I_x = \frac{b^3t}{3}.$$

$$\text{Max. } S_s = \frac{3}{2\sqrt{2}}\frac{P}{bt} = 1.0605\frac{P}{bt}.$$

If the legs of an angle section are vertical and horizontal, and if the small amount of shear carried by the horizontal leg is neglected, the shear center is at the intersection of the median lines of the two legs parallel to the longer dimensions.

EXAMPLE

A 3- by 3- by ¼-in. angle section used as a cantilever beam with the axis of symmetry of the cross section horizontal carries a load of 500 lb. Compute the maximum unit shearing stress.

Solution:

$$b = 2.875 \text{ in.} \qquad t = 0.25 \text{ in.}$$

$$\text{Max. } S_s = \frac{1.0605 \times 500}{2.875 \times 0.25} = 738 \text{ lb./in.}^2.$$

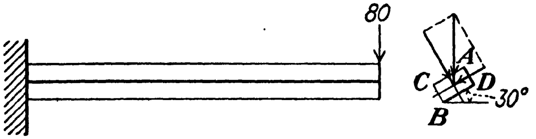
PROBLEMS

1. A 6- by 6- by ½-in. angle section 5 ft. long is used as a cantilever beam and is placed with the axis of symmetry horizontal. If loaded with a load at the free end to cause a maximum unit flexural stress of 18,000 lb./in.^{2}, what is the maximum unit shearing stress? *Ans.* 830 lb./in.^{2}.

2. Check the value of I_x as given by the expression derived above for an 8- by 8- by ½-in. angle section with that obtained from values given in Table XI.

71. Stresses Due to Oblique Loads.—If the cross section of a beam has two axes of symmetry, the shear center is at the centroid of the section. If such a beam has a load applied through the centroid of the section, but at an angle α with one of the principal axes, it is necessary to resolve the load into two rectangular components acting along the principal axes. The stresses at any point due to these component loads are then computed separately and added algebraically.

If the cross section of a beam has only one axis of symmetry or no axis of symmetry and if there is to be no twist, it is necessary for the load to act through the shear center as already discussed. If such a load is not acting normal to one of the principal axes, it is necessary to resolve the load at the shear center into two rectangular components. One of these components must be along the principal axis which passes through the shear center, and the other component parallel to the other principal axis. The values of the stresses are then computed as above.



(a) (b)

FIG. 114.

EXAMPLE 1

A cantilever beam 4 in. wide, 2 in. deep, and 4 ft. long is inclined at an angle of 30° with the horizontal, as shown in Fig. 114. Compute the stress at the top and bottom corners at the fixed end due to a load of 80 lb. applied vertically at the free end.

Solution.—The force of 80 lb. is resolved into two rectangular components, one of $80 \times 0.866 = 69.3$ lb. along the shorter principal axis, and one of $80 \times 0.50 = 40$ lb. along the longer principal axis. The stress S_1 due to the component of 69.3 lb. is given by the expression,

$$S_1 = \frac{69.3 \times 48 \times 6}{4 \times 2 \times 2} = 1247 \text{ lb./in.}^2.$$

The stress S_2 due to the component of 40 lb. is given by the expression,

$$S_2 = \frac{40 \times 48 \times 6}{2 \times 4 \times 4} = 360 \text{ lb./in.}^2.$$

The total stress at the top corner A is $1247 + 360 = 1607$ lb./in.^{2} tension, and the stress at the bottom corner B is the same amount in compression. At the corner C the stress is $1247 - 360 = 887$ lb./in.^{2} in tension, and at the corner D the stress is the same in compression.

EXAMPLE 2

A 6- by 4- by 1-in. angle bar 10 ft. long is supported at the ends with the long leg vertical and loaded with a concentrated load of 4000 lb. acting through the shear center at the middle of the beam. Neglecting the fillet and rounded corners, compute the stress at each corner of each leg at the middle.

Solution.—The cross section of the beam is shown in Fig. 115. By the principles of mechanics,

$$\tan 2\alpha = 1.$$
$$2\alpha = 45°.$$
$$\alpha = 22° 30'.$$

The load of 4000 lb. is resolved into its two rectangular components, 4000 × cos 22° 30′ = 3695 lb. parallel to axis 3-3, and 4000 × sin 22° 30′ = 1530 lb. parallel to axis 4-4.

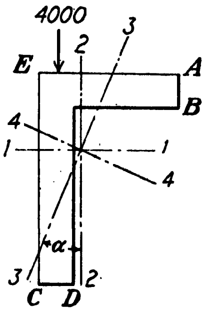


FIG. 115.

$$I_{3\text{-}3} = 6.5 \text{ in.}^4, \text{ and } I_{4\text{-}4} = 35.1 \text{ in.}^4.$$

With respect to axis 4-4, the moment at the middle of the beam is

$$M_1 = 3695 \times \frac{120}{4} = 110{,}850 \text{ in.-lb.}$$

For axis 4-4, the distance $c_A = 3.08$ in., $c_B = 2.16$ in., $c_C = 3.98$ in., and $c_D = 3.60$ in. The stresses at the four corners due to the moment M_1 are as follows:

$$\text{At } A,\ S_1 = 110{,}850 \times \frac{3.08}{35.1} = 9740 \text{ lb./in.}^2 \text{ C.}$$
$$\text{At } B,\ S_1 = 110{,}850 \times \frac{2.16}{35.1} = 6810 \text{ lb./in.}^2 \text{ C.}$$
$$\text{At } C,\ S_1 = 110{,}850 \times \frac{3.98}{35.1} = 12{,}570 \text{ lb./in.}^2 \text{ T.}$$
$$\text{At } D,\ S_1 = 110{,}850 \times \frac{3.60}{35.1} = 11{,}350 \text{ lb./in.}^2 \text{ T.}$$

With respect to axis 3-3, the moment at the middle of the beam is

$$M_2 = 1530 \times \frac{120}{4} = 45{,}900 \text{ in.-lb.}$$

For axis 3-3, the distance $c_A = 1.78$ in., $c_B = 2.17$ in., $c_C = 0.38$ in., and $c_D = 1.30$ in. The stresses at the four corners due to the moment M_2 are as follows:

$$\text{At } A,\ S_2 = 45{,}900 \times \frac{1.78}{6.5} = 12{,}600 \text{ lb./in.}^2 \text{ T.}$$
$$\text{At } B,\ S_2 = 45{,}900 \times \frac{2.17}{6.5} = 15{,}320 \text{ lb./in.}^2 \text{ T.}$$

$$\text{At } C,\ S_2 = 45{,}900 \times \frac{0.38}{6.5} = 2680 \text{ lb/in.}^2 \text{ T.}$$

$$\text{At } D,\ S_2 = 45{,}900 \times \frac{1.30}{6.5} = 9180 \text{ lb./in.}^2 \text{ T.}$$

At each corner the stress is the algebraic sum of the two stresses.

$$\begin{aligned} S_A &= 12{,}600 - 9740 = 2860 \text{ lb./in.}^2 \text{ T.} \\ S_B &= 15{,}320 - 6810 = 8510 \text{ lb./in.}^2 \text{ T.} \\ S_C &= 12{,}570 + 2680 = 15{,}250 \text{ lb./in.}^2 \text{ T.} \\ S_D &= 11{,}350 + 9180 = 20{,}530 \text{ lb./in.}^2 \text{ T.} \end{aligned}$$

PROBLEMS

1. Solve Example 1 above if the surface AC of the beam is inclined at an angle of 15° with the horizontal.

Ans. $S_A = 1576$ lb./in.^{2} T.; $S_D = 1204$ lb./in.^{2} C.

2. Compute the stress at the corner E in the angle section shown in Fig. 115. *Ans.* 18,380 lb./in.^{2} C.

3. Wooden beams 4 in. wide, 6 in. deep, and 10 ft. long, supported at the ends, are used as purlins (longitudinal beams) on a roof which has a slope of 1 vertical to 3 horizontal. If the allowable flexural stress is 1200 lb./in.^{2}, what is the total uniformly distributed load each purlin can support?

Ans. 1350 lb.

4. A Z-bar section is 6 in. deep with flanges 3½ in. wide. Both flanges and web are ¾ in. thick. If it is to be used as a cantilever beam 8 ft. long with the web and the load vertical, and the allowable unit stress $S = 18{,}000$ lb./in.^{2}, what concentrated load can be applied at the end? Neglect fillets and rounded corners. *Ans.* 1030 lb.

72. Deflections Due to Oblique Loads.—If the load on a beam is applied through the shear center but at an angle with one of the principal axes of the cross section other than 90°, it is necessary to resolve it at the shear center into components acting along or parallel to the principal axes. The deflections in the two directions are then computed separately and combined vectorially to obtain the resultant deflection.

If the load is not applied through the shear center, a torsional effect is caused as well as the flexural effect.

EXAMPLE

A simple wooden beam 6 in. wide, 2 in. deep, and 10 ft. long is supported with the ends on the same level, but with the 6-in. face at an angle of 30° with the horizontal. If $E = 1{,}500{,}000$ lb./in.^{2}, what will be the amount and direction of the deflection due to a vertical load of 200 lb. at the middle?

Solution.—The load of 200 lb. is resolved into its two components along the principal axes of inertia of the cross section. The component normal to the 6-in. face is 200 × 0.866 = 173 lb. The component normal to the 2-in. face is 200 × 0.5 = 100 lb.

The deflection in the direction normal to the 6-in. face is

$$y_1 = \frac{Pl^3}{48EI} = \frac{173 \times 120^3 \times 12}{48 \times 1{,}500{,}000 \times 6 \times 8} = 1.038 \text{ in.}$$

The deflection in the direction normal to the 2-in. face is

$$y_2 = \frac{Pl^3}{48EI} = \frac{100 \times 120^3 \times 12}{48 \times 1{,}500{,}000 \times 2 \times 216} = 0.067 \text{ in.}$$

$$y = \sqrt{y_1^2 + y_2^2} = 1.04 \text{ in.}$$

The angle of this resultant deflection with the direction of y_1 is given by the expression.

$$\theta = \tan^{-1}\frac{0.067}{1.038} = 3^\circ\ 41'.$$

The angle of the resultant deflection with the vertical is therefore 30° − 3° 41′ = 26° 19′.

PROBLEMS

1. Solve the example above if the 6-in. face is at an angle of 45° with the horizontal. *Ans.* 0.854 in.; 38° 40′ with the vertical.

2. Compute the deflection of the beam described in Problem 4, Art. 71, using $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 0.866 in.; 51° 15′ with the vertical.

3. A 6- by 6- by 1-in. rolled angle section 20 ft. long is supported at the ends with the legs vertical and horizontal, and is loaded with a uniformly distributed load of 60 lb. per linear ft. including its own weight. Compute the deflection, using $E = 29{,}000{,}000$ lb./in.^{2}.

Ans. 0.368 in.; 30° 10′ with the vertical.

CHAPTER VIII

FIXED AND CONTINUOUS BEAMS

73. Beam Fixed at Both Ends, Load Uniformly Distributed.—Figure 116(*a*) represents a beam fixed at both ends and loaded with a uniformly distributed load of w per unit length. Figure 116(*b*) shows a portion at the left end x units long taken as a free body. By the symmetry of the entire beam, the shear $V_o = wl/2$.

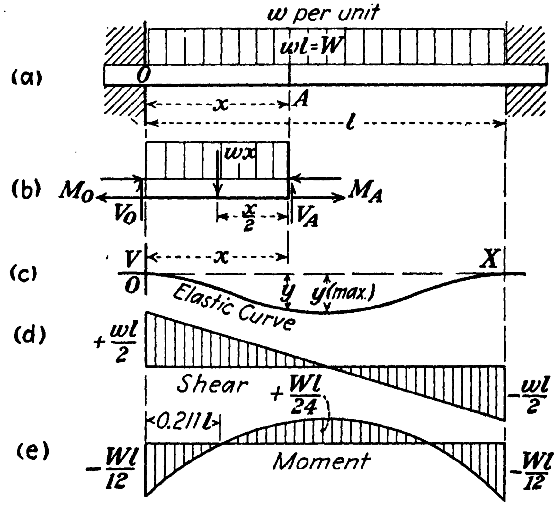


FIG. 116.

Let the origin be taken at O, the left end of the beam. The differential equation becomes

$$EI\frac{d^2y}{dx^2} = M_o + \frac{wlx}{2} - \frac{wx^2}{2}.$$

By integration,

$$EI\frac{dy}{dx} = M_o x + \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1.$$

Since $dy/dx = 0$ where $x = 0$, $C_1 = 0$.

Also, since $dy/dx = 0$ where $x = l$,

$$0 = M_o l + \frac{wl^3}{4} - \frac{wl^3}{6}.$$

$$M_o = -\frac{wl^2}{12} = -\frac{Wl}{12}.$$

$$EI\frac{dy}{dx} = -\frac{wl^2x}{12} + \frac{wlx^2}{4} - \frac{wx^3}{6}.$$

$$EIy = -\frac{wl^2x^2}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24} + C_2.$$

Since $y = 0$ where $x = 0$, $C_2 = 0$.

$$EIy = -\frac{wl^2x^2}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24}.$$

By symmetry, the maximum value of y is at the middle.

$$\text{Max. } y = -\frac{Wl^3}{384EI}.$$

PROBLEMS

1. Derive the expression for the moment at the middle of the beam.

Ans. $M = +\frac{Wl}{24}$.

2. Locate the points of counterflexure for which $M = 0$.

Ans. $x = 0.211l$ or $0.789l$.

3. A circular steel rod 3 in. in diameter and 25 ft. long is fixed at both ends. Compute the maximum stress due to its own weight. Using E as 30,000,000 lb./in.², compute the deflection. *Ans.* 5660 lb./in.²; 0.354 in.

4. A 24-in. 100-lb. I-beam 40 ft. long is fixed at both ends. If the allowable flexural stress is 18,000 lb./in.², what additional load per linear foot may be added? If E = 29,000,000 lb./in.², what is the maximum deflection?

Ans. 2125 lb./ft.; 0.373 in.

5. Derive the expression for the maximum deflection of the fixed beam with uniformly distributed load in terms of the maximum stress.

Ans. $y = Sl^2/32Ec$.

6. A 10-in. 40-lb. I-beam 25 ft. long is fixed at both ends and loaded with a uniformly distributed load, including its own weight, to cause a maximum bending stress of 18,000 lb./in.². Using E as 29,000,000 lb./in.², compute the total superimposed load required and the maximum deflection.

Ans. 21,750 lb.; 0.349 in.

74. Beam Fixed at Both Ends, Concentrated Load at Middle.—Figure 117(*a*) represents a beam fixed at both ends with a concentrated load P at the middle. Figure 117(*b*) shows a portion of the beam at the left end x units long taken as a free body. By the symmetry of the entire beam the shear $V_o = P/2$.

Let the origin be taken at O, the left end of the beam. The differential equation becomes

$$EI\frac{d^2y}{dx^2} = M_o + \frac{Px}{2}.$$

By integration,

$$EI\frac{dy}{dx} = M_ox + \frac{Px^2}{4} + C_1.$$

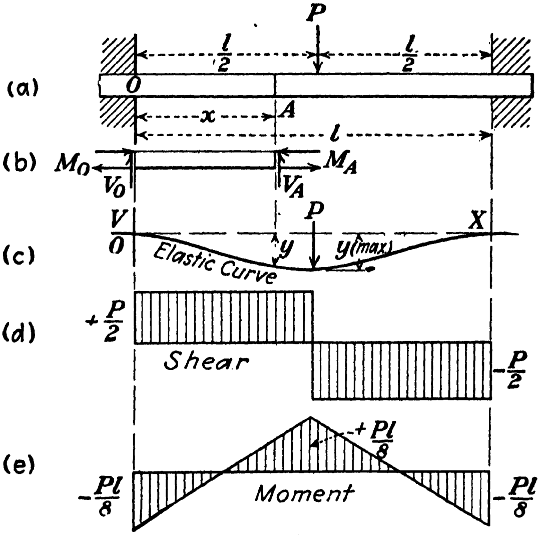


FIG. 117.

Since $dy/dx = 0$ where $x = 0$, $C_1 = 0$.

$$EI\frac{dy}{dx} = M_ox + \frac{Px^2}{4}.$$

Also, $dy/dx = 0$ where $x = l/2$.

$$0 = \frac{M_ol}{2} + \frac{Pl^2}{16}.$$

$$M_o = -\frac{Pl}{8}.$$

$$EI\frac{dy}{dx} = -\frac{Plx}{8} + \frac{Px^2}{4}.$$

$$EIy = -\frac{Plx^2}{16} + \frac{Px^3}{12} + C_2.$$

Since $y = 0$ where $x = 0$, $C_2 = 0$.

$$EIy = -\frac{Plx^2}{16} + \frac{Px^3}{12}.$$

This is the equation of the elastic curve of the left half of the beam. The right half of the beam is symmetrical with the left half.

The maximum deflection is at the middle and its amount is given by the expression,

$$\text{Max. } y = -\frac{Pl^3}{192EI}.$$

For a single moving load, both the maximum stress and the maximum deflection occur when the load is at the middle.

PROBLEMS

1. Derive the expression for the moment at the middle of the beam.
Ans. $M = +\frac{Pl}{8}$.

2. Locate the points of counterflexure for which $M = 0$.
Ans. $x = \frac{1}{4}l$ and $\frac{3}{4}l$.

3. Derive the expression for the maximum deflection of the fixed beam with a concentrated load at the middle in terms of the maximum flexural stress.
Ans. $y = Sl^2/24Ec$.

4. A steel rod 1 in. in diameter and 12 ft. long is fixed at both ends. Compute the concentrated load at the middle which will cause a maximum bending stress of 16,000 lb./in.². Using $E = 30{,}000{,}000$ lb./in.², compute the maximum deflection.
Ans. 87 lb.; 0.922 in.

5. A wooden beam 2 in. wide, 10 in. deep, and 18 ft. long is fixed at both ends. Compute the bending stress due to a load of 1200 lb. at the middle. Using $E = 1{,}400{,}000$ lb./in.², compute the maximum deflection.
Ans. 972 lb./in.²; 0.27 in.

6. A 5-in. 10-lb. I-beam 15 ft. long is fixed at both ends. Compute the load at the middle which will cause a deflection of ¼ in., if $E = 29{,}000{,}000$ lb./in.². Compute the maximum bending stress.
Ans. 2888 lb.; 13,420 lb./in.².

75. Beam Fixed at One End, Supported at the Other, Load Uniformly Distributed.—Figure 118(*a*) represents a beam which is fixed at the right end, supported at the left end on the same level, and loaded with a uniformly distributed load of w per unit length. Let the origin O be taken at the left end of the beam. The differential equation becomes

$$EI\frac{d^2y}{dx^2} = Rx - \frac{wx^2}{2}.$$

By integration,

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{wx^3}{6} + C_1.$$

At the right end, $dy/dx = 0$ and $x = l$.

$$0 = \frac{Rl^2}{2} - \frac{wl^3}{6} + C_1.$$

$$C_1 = -\frac{Rl^2}{2} + \frac{wl^3}{6}.$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{Rl^2}{2} - \frac{wx^3}{6} + \frac{wl^3}{6}.$$

$$EIy = \frac{Rx^3}{6} - \frac{Rl^2x}{2} - \frac{wx^4}{24} + \frac{wl^3x}{6} + C_2.$$

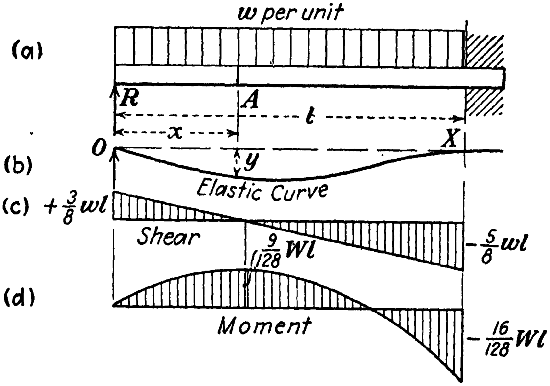


FIG. 118.

At the left end, $y = 0$ and $x = 0$, so $C_2 = 0$.

$$EIy = \frac{Rx^3}{6} - \frac{Rl^2x}{2} - \frac{wx^4}{24} + \frac{wl^3x}{6}.$$

At the right end, $y = 0$ and $x = l$.

$$0 = \frac{Rl^3}{6} - \frac{Rl^3}{2} - \frac{wl^4}{24} + \frac{wl^4}{6}.$$

$$R = \frac{3}{8}wl.$$

The equation of the elastic curve becomes

$$EIy = \frac{wlx^3}{16} - \frac{wl^3x}{48} - \frac{wx^4}{24}.$$

The equation of the slope becomes

$$EI\frac{dy}{dx} = \frac{3wlx^2}{16} - \frac{wx^3}{6} - \frac{wl^3}{48}.$$

To locate the point of maximum deflection, the equation for the slope is equated to zero, from which the following equation is obtained:

$$8x^3 - 9lx^2 + l^3 = 0.$$

The slope is zero at the fixed end, so $x - l = 0$ must be one of the factors of this equation. The remaining quadratic is

$$8x^2 - lx - l^2 = 0.$$

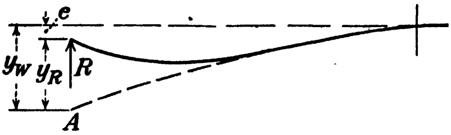


FIG. 119.

The positive root of this equation is

$$x = 0.4215l.$$

The amount of the maximum deflection is obtained by substituting this value of x in the equation of the elastic curve.

With the reaction R known, the shear diagram may be drawn. From the shear diagram, the moment and the stress at any point may be computed.

If the support at the left end of the beam settles a distance e below the level of the fixed end, the reaction R at the left end is decreased, while if it is raised above the level of the fixed end it is increased. If there were no support at the left end, the beam would deflect an amount y_W to point A, Fig. 119. Since the reaction R varies directly as the distance above point A where it is zero, and since $R = \frac{3}{8}W$ when the left end is on the same level as the fixed end, the value of R when the supported end has settled an amount e, is given by the relation,

$$R = \frac{y_W - e}{y_W}\left(\frac{3}{8}W\right).$$

If the supported end is raised above the level of the fixed end, the value of e is negative.

If a beam is supported at the ends and at the middle, and is loaded with a uniformly distributed load over both spans, as shown in Fig. 120, each half of the beam is under the same conditions of loading and support as the beam discussed above. The weight of each half of the beam is wl, or W.

If the middle support settles a distance e below the level of the end supports, the condition of each half of the beam is the same as that of the beam fixed at one end and supported at the other, the supported end being higher than the fixed end. If both end supports settle below the level of the middle support, the condition of each half of the beam is the same as that of the beam fixed at one end and supported at the other, the supported end being lower than the fixed end. If only one end support settles below the level of the other two, the case is approximately the same as the preceding, the distance e, however, being only one-half of the settlement.

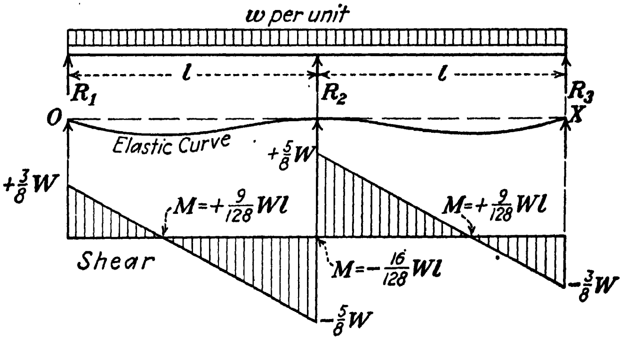


Fig. 120.

EXAMPLE 1

A 10-in. 40-lb. I-beam 24 ft. long is fixed at one end, supported at the other on the same level, and loaded with a uniformly distributed load of 600 lb. per linear ft. including its own weight. Compute the maximum stress.

Solution:

$$W = 600 \times 24 = 14{,}400 \text{ lb.} \qquad R = \tfrac{3}{8}W = 5400 \text{ lb.}$$

The maximum moment is at the fixed end.

$$M = 5400 \times 24 - 14{,}400 \times 12 = 43{,}200 \text{ ft.-lb.}$$

$$S = 43{,}200 \times \frac{12}{31.6} = 16{,}400 \text{ lb./in.}^2$$

EXAMPLE 2

Solve Example 1 if the supported end settles 0.6 in. Use $E = 29{,}000{,}000$ lb./in.^{2}.

Solution.—If there were no support at the left end, the deflection would be

$$y_W = \frac{Wl^3}{8EI} = \frac{14{,}400 \times 288^3}{8 \times 29{,}000{,}000 \times 158} = 9.384 \text{ in.}$$

$$y_R = y_W - e = 8.784 \text{ in.}$$

$$R = \frac{8\,784}{9.384} \times 5400 = 5054 \text{ lb.}$$

The maximum moment is at the fixed end.

$$M = 5054 \times 24 - 14{,}400 \times 12 = 51{,}504 \text{ ft.-lb.}$$

$$S = 51{,}504 \times \frac{12}{31.6} = 19{,}560 \text{ lb./in.}^2.$$

PROBLEMS

1. Compute the value of the maximum deflection of the beam which is fixed at one end, supported on the same level at the other, and loaded with a uniformly distributed load. *Ans.* $y = -0.00543\dfrac{Wl^3}{EI}$.

2. A wooden beam 4 in. wide, 10 in. deep, and 15 ft. long is fixed at one end, supported at the other on the same level, and loaded with a uniformly distributed load of 300 lb. per linear ft. Compute the maximum bending stress. Using E as 1,200,000 lb./in.^{2}, compute the maximum deflection. *Ans.* 1520 lb./in.^{2}; 0.356 in.

3. Compute the maximum bending stress in the beam described in Problem 2 if the supported end settles ¼ in. *Ans.* 1660 lb./in.^{2}.

4. Compute the maximum bending stress in the beam described in Problem 2 if the supported end is raised 1 in. above the level of the fixed end. *Ans.* 1075 lb./in.^{2}.

5. Solve for the reactions of the beam shown in Fig. 120 by the following method. Write the expression for the deflection at the middle if the middle support is removed. Solve for the upward force R_2 at the middle necessary to reduce this deflection to zero.

6. Prove that a uniformly loaded beam extending over three equally spaced supports on the same level does not have its strength affected if cut through at the middle support.

7. A 10-in. 40-lb. I-beam 40 ft. long is supported at the ends and at the middle. If allowable $S = 18{,}000$ lb./in.^{2}, what superimposed load per foot can it carry if the supports are on the same level? *Ans.* 908 lb./ft.

8. Compute the maximum stress in the beam described in Problem 7 if the middle support settles 1 in. *Ans.* 13,150 lb./in.^{2}.

76. Beam Fixed at One End, Supported at the Other, Load Concentrated.—Figure 121 represents a beam fixed at the right end, supported at the left end on the same level, and loaded with a concentrated load P at a distance a from the left end and a distance b from the right end. If there were no support R, the deflection of the end, as shown in Art. 63, would be

$$y_P = -\frac{Pb^2}{EI}\left(\frac{a}{2} + \frac{b}{3}\right).$$

The upward pressure R at the free end of the beam necessary to deflect the end of the beam a distance y, as shown in Art. 59, is

$$y_R = \frac{Rl^3}{3EI}.$$

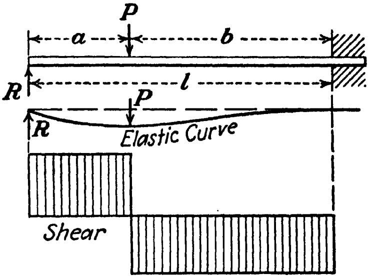


FIG. 121.

In order that the supported end may be brought to the same level as the fixed end, these two values of y must be equal numerically:

$$\frac{Rl^3}{3EI} = \frac{P}{EI}\left(\frac{b^3}{3} + \frac{ab^2}{2}\right).$$

$$R = \frac{Pb^2}{l^3}(b + 1.5a).$$

With the reaction R known, the moment at any section in the beam may be computed. If the complete variation in the moment is desired, the shear diagram may be drawn and the moment computed from its area.

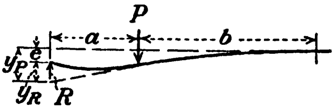


FIG. 122.

The deflection of the beam at any point may be computed from the algebraic sum of the deflections caused by the load P and the reaction R separately.

If the equations of the elastic curves are desired, the same method of solution is used as that in Art. 65.

If the support at the left end of the beam settles a distance e below the level of the fixed end, as shown in Fig. 122, the reaction at the supported end will be y_R/y_P times the amount the reaction would be if it were on the same level.

If a beam is supported at the ends and at the middle, and is loaded with two concentrated loads placed symmetrically, each half of the beam is under the same conditions of loading and support as the beam discussed above.

PROBLEMS

1. Compute the reaction at the supported end, the moment under the load, and the moment at the fixed end if the load is at the middle.

Ans. $\frac{5}{16}P$; $\frac{5}{32}Pl$; $-\frac{6}{32}Pl$.

2. Compute the relation between a and b to give the same numerical value of the moment under the load and at the fixed end.

Ans. $a = 0.707b$. ($a = 0.414l$; $b = 0.586l$.)

3. A 12-in. 50-lb. I-beam 30 ft. long is supported at the ends and at the middle. It carries a 10,000-lb. load 5 ft. from the left end and another of the same amount 5 ft. from the right end. Compute the maximum stress due to the concentrated loads if the supports are on the same level.

Ans. 6180 lb./in.^{2}.

4. Solve Problem 3 if the middle support settles 1 in. and $E = 29{,}000{,}000$ lb./in.^{2}.

Ans. 11,550 lb./in.^{2}.

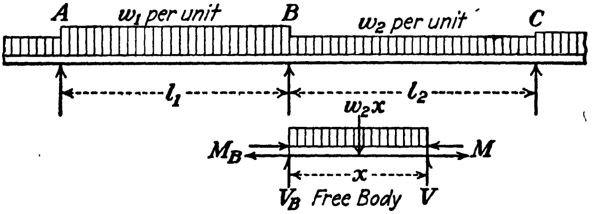


FIG. 123.

77. Theorem of Three Moments, Load Uniformly Distributed.

If a beam is continuous over four or more supports, the method of writing the differential equation for each span and determining the constants of integration and the unknown moments and shears becomes very tedious. The theorem of three moments, however, gives the relation between the moments at three successive supports, from which the reactions and stresses may be obtained very easily. The theorem of three moments will not determine the deflections, but these are seldom required.

Figure 123 represents two adjoining spans of a continuous beam which extends indefinitely in each direction. The beam is loaded with a uniformly distributed load of w_1 per unit length over the span of length l_1 and with a load of w_2 per unit length over the next span of length l_2. The supports are all on the same level. For the free body shown, which consists of a length of x units at the left end of the second span,

$$M = M_B + V_B x - \frac{w_2 x^2}{2}. \tag{1}$$

The differential equation becomes

$$EI\frac{d^2y}{dx^2} = M_B + V_B x - \frac{w_2 x^2}{2}.$$

$$EI\frac{dy}{dx} = M_B x + \frac{V_B x^2}{2} - \frac{w_2 x^3}{6} + C_1.$$

If $x = 0$, $C_1 = EI\frac{dy}{dx}$, $\frac{dy}{dx}$ being the slope at B.

$$EIy = \frac{M_B x^2}{2} + \frac{V_B x^3}{6} - \frac{w_2 x^4}{24} + C_1 x + C_2.$$

At reaction B, $y = 0$ and $x = 0$, so $C_2 = 0$.
At reaction C, $y = 0$ and $x = l_2$, so

$$\frac{M_B l_2}{2} + \frac{V_B l_2^2}{6} - \frac{w_2 l_2^3}{24} + C_1 = 0.$$

In Equation (1), if $x = l_2$, M becomes M_C.

$$M_C = M_B + V_B l_2 - \frac{w_2 l_2^2}{2}.$$

By eliminating V_B between the last two equations,

$$2M_B l_2 + M_C l_2 + \frac{w_2 l_2^3}{4} + 6C_1 = 0.$$

A similar expression may be written for the first span, with the origin at point B and using x positive to the left.

$$2M_B l_1 + M_A l_1 + \frac{w_1 l_1^3}{4} + 6C_3 = 0.$$

At support B, $C_3 = EI\frac{dy}{dx}$, with x positive to the left. C_3 is therefore equal to $-C_1$. If C_3 is replaced by $-C_1$ and the two equations are added, the equation of three moments is obtained.

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}.$$

In case the spans are equal and the loads are equal, the special theorem of three moments becomes

$$M_A + 4M_B + M_C = -\frac{wl^2}{2}.$$

78. Computation of Moments.—The equations derived in Art. 77 give the relation between the moments at any three successive supports. In making solution, an equation is written for supports 1, 2, and 3, another for supports 2, 3, and 4, and so on until the last support is used. There will then be two less equations than the number of unknown moments. If the beam is not fixed at the ends, the moments at the first and last supports are known. If the support is at the end of the beam, the moment at the support is zero. If the beam overhangs the end support, the moment at the support is easily computed by static conditions, the overhanging part being used as the free body.

If the continuous beam is fixed at both ends and has equal spans and equal loads, each span is under the same condition as a beam fixed at both ends. The moment at each end and at each support is then $-wl^2/12$. The moment at the middle of each span is $+wl^2/24$.

EXAMPLE 1

A beam 46 ft. long has four equal spans of 10 ft. and an overhang of 6 ft on the right end, as shown in Fig. 124. If the beam weighs 100 lb. per linear ft., compute the moment at each support.

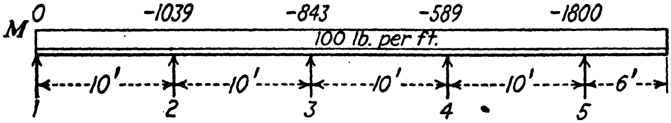


FIG. 124.

Solution.—Since the first support is at the left end, $M_1 = 0$. The part on the right of support 5 weighs 600 lb., so

$$M_5 = -600 \times 3 = -1800 \text{ ft.-lb.}$$

By the theorem of three moments for equal spans and equal loads,

$$0 + 4M_2 + M_3 = -5000.$$
$$M_2 + 4M_3 + M_4 = -5000.$$
$$M_3 + 4M_4 - 1800 = -5000.$$

By the solution of these three equations,

$$\begin{aligned} M_2 &= -1039 \text{ ft.-lb.} \\ M_3 &= -843 \text{ ft.-lb.} \\ M_4 &= -589 \text{ ft.-lb.} \end{aligned}$$

EXAMPLE 2

A beam 60 ft. long has spans of 20 ft., 30 ft., and 10 ft. which carry loads of 500 lb., 200 lb., and 800 lb. per linear ft., respectively. Compute the moment at each support.

Solution.—M_1 and M_4 are both zero. By the theorem of three moments for unequal spans and unequal loads,

$$\begin{aligned} 0 + 100M_2 + 30M_3 &= -2{,}350{,}000. \\ 30M_2 + 80M_3 + 0 &= -1{,}550{,}000. \end{aligned}$$

By the solution of these two equations,

$$\begin{aligned} M_2 &= -19{,}830 \text{ ft.-lb.} \\ M_3 &= -11{,}900 \text{ ft.-lb.} \end{aligned}$$

PROBLEMS

1. A beam 60 ft. long has four equal spans and carries a load of 300 lb. per linear ft. over the entire length. Compute M_2 and M_3.

Ans. $M_2 = -7233$ ft.-lb.; $M_3 = -4821$ ft.-lb.

2. Solve for the moments at the remaining supports if the first and third supports are removed from the beam described in Problem 1.

Ans. $M_1 = -33{,}750$ ft.-lb.; $M_2 = -14{,}062$ ft.-lb.

3. A beam 50 ft. long has two end spans of 20 ft. each and a middle span of 10 ft. The two end spans carry loads of 400 lb. per linear ft. and the center span carries a load of 600 lb. per linear ft. Compute M_2.

Ans. $-13{,}570$ ft.-lb.

4. A beam 48 ft. long is supported at the left end, at 20 ft. from the left end, and at 40 ft. from the left end. Get M_2 and M_3 due to a load of 600 lb. per linear ft. *Ans.* $-25{,}200$ ft.-lb.; $-19{,}200$ ft.-lb.

5. For a beam with two equal spans of length l and loaded with a uniformly distributed load of w per unit, what must be the overhang l_1 outside of reactions 1 and 3 so that the moments at all three supports are equal?

Ans. $0.408l$.

6. A steel shaft 3 in. in diameter and 50 ft. long has three equal spans of 15 ft. each, and an overhang of 2.5 ft. at each end. At the left end is a pulley weighing 100 lb. on which is a downward belt pull of 250 lb. At the right end is a pulley weighing 100 lb. on which is an upward belt pull of 280 lb. Compute the moment at each support.

Ans. $M_1 = -950$ ft.-lb.; $M_2 = -262$ ft.-lb.; $M_3 = -703$ ft.-lb.; $M_4 = +375$ ft.-lb.

7. A beam 100 ft. long and carrying a uniformly distributed load of 800 lb. per linear ft. is supported at the left end, and at 40 ft., 70 ft., and 90 ft from the left end. Compute the moment at each reaction.

Ans. $M_1 = 0$; $M_2 = -124{,}730$ ft.-lb.; $M_3 = -24{,}580$ ft.-lb.; $M_4 = -40{,}000$ ft.-lb.

8. A beam 80 ft. long and carrying a uniformly distributed load of 480 lb. per linear ft. is fixed at both ends and has four equal spans. Compute the moments at the ends and at each support. *Ans.* $M = -16{,}000$ ft.-lb.

79. Computation of Reactions.—In order to determine the reactions of the supports of a continuous beam uniformly loaded, the moments at the supports are computed first, as explained in Art. 78. With the moments known, the first span at either end, including the overhang, if any, may be taken as a free body to determine the end reaction. The first and second spans together may next be taken as a free body to determine the second reaction and so on as far as necessary. When all but two of the reactions have been determined, the entire beam may be taken as the free body. The equation $\Sigma M = 0$ with respect to a center of moments on one of the reactions will determine the other. The equation $\Sigma F_y = 0$ will determine the last reaction. If the support and loading of a beam are symmetrical, the corresponding reactions are equal. If a continuous beam with equal spans and equal loads is fixed at both ends, the vertical shear at each end is $wl/2$, and each of the intermediate reactions is wl.

After the reactions are computed, the shear diagram may be drawn and the maximum positive moments computed. In general, the controlling moment will be at one of the supports.

EXAMPLE 1

Compute the moments and the reactions of the beam shown in Fig. 125

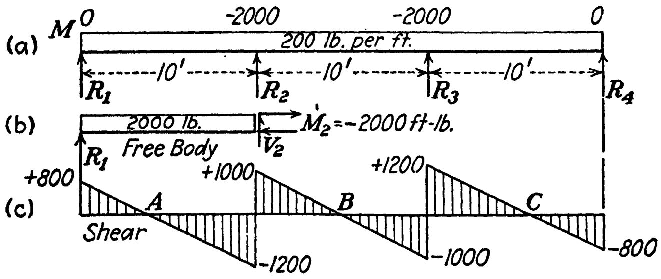


FIG. 125.

Solution.—M_1 and M_4 are both zero.

$$0 + 4M_2 + M_3 = -\frac{200 \times 100}{2}.$$

By symmetry,

$$M_2 = M_3 = -2000 \text{ ft.-lb.}$$

By symmetry, $R_1 = R_4$ and $R_2 = R_3$.

Figure 125(*b*) shows the first span at the left taken as a free body, the section being made just at the left of reaction R_2. The shear V_2 at the section is unknown. The moment M_2 is known and acts in the direction shown. Since the free body is necessarily in equilibrium, the summation of all of the moments of the forces on the free body with respect to any point is equal to zero. With the right end of the free body as the center of moments, the moment equation is

$$\begin{aligned} 10R_1 + 2000 &= 2000 \times 5. \\ R_1 &= 800 \text{ lb.} \end{aligned}$$

By symmetry, R_4 is also 800 lb.
The equation $\Sigma F_y = 0$ gives

$$\begin{aligned} R_2 + R_3 + 1600 &= 6000. \\ R_2 + R_3 &= 4400 \text{ lb.} \end{aligned}$$

By symmetry, $R_2 = R_3 = 2200$ lb.
The shear V_2 may now be computed.

$$V_2 = 800 - 2000 = -1200 \text{ lb.}$$

The shear diagram, Fig. 125(*c*), may now be drawn, and the moments at the other three danger sections computed.

$$\begin{aligned} M_A &= M_C = +1600 \text{ ft.-lb.} \\ M_B &= +500 \text{ ft.-lb.} \end{aligned}$$

The controlling moment is at either reaction R_2 or R_3.

EXAMPLE 2

Compute the moments and the reactions of the beam shown in Fig. 126.

Solution.—M_1 and M_4 are both zero. By the general equation of three moments,

$$\begin{aligned} 0 + 100M_2 + 25M_3 &= -\frac{300 \times 25^3}{4} - \frac{400 \times 25^3}{4}. \\ 4M_2 + M_3 &= -109{,}375. \\ 25M_2 + 150M_3 + 0 &= -\frac{400 \times 25^3}{4} - \frac{200 \times 50^3}{4}. \\ M_2 + 6M_3 &= -312{,}500. \\ 24M_2 + 6M_3 &= -656{,}250. \\ 23M_2 &= -343{,}750. \\ M_2 &= -14{,}950 \text{ ft.-lb.} \\ M_3 &= -49{,}590 \text{ ft.-lb.} \end{aligned}$$

The left span is shown as a free body in Fig. 126(*b*). The moment equation, with the center of moments on the line of action of V_2, gives

$$25R_1 + 14{,}950 = 7500 \times 12.5.$$
$$R_1 = 3152 \text{ lb.}$$

The right span is shown as a free body in Fig. 126(*c*). The moment equation, with the center of moments on the line of action of V_3 gives

$$50R_4 + 49{,}590 = 250{,}000.$$
$$R_4 = 4008 \text{ lb.}$$

There are now only two unknown reactions. With the entire beam as the free body, the moment equation with respect to an axis on reaction R_3 gives

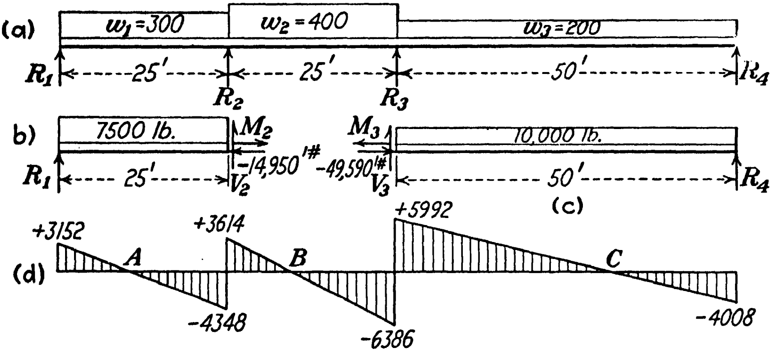


FIG. 126.

$$25R_2 + 3152 \times 50 - 4008 \times 50 - 7500 \times 37.5 - 10{,}000 \times 12.5 + 10{,}000 \times 25 = 0.$$
$$R_2 = 7962 \text{ lb.}$$

The equation $\Sigma F_y = 0$ gives

$$R_3 = 27{,}500 - 15{,}122 = 12{,}378 \text{ lb.}$$

The shear diagram is shown in Fig. 126(*d*). The moment at *A* is +16,560 ft.-lb. That at *B* is very small, being only +1376 ft.-lb. That at *C* is +40,160 ft.-lb. The maximum moment in the beam is that at R_3, −49,590 ft.-lb.

PROBLEMS

1. Compute the reactions of the beam described in Problem 1, Art. 78.
Ans. $R_1 = R_5 = 1768$ lb.; $R_2 = R_4 = 5143$ lb.; $R_3 = 4178$ lb.

2. Compute the reactions of the beam described in Problem 7, Art. 78.
Ans. $R_1 = 12{,}882$ lb.; $R_2 = 34{,}456$ lb.; $R_3 = 15{,}892$ lb.; $R_4 = 16{,}771$ lb,

3. A beam 40 ft. long with four equal spans of 10 ft. each carries a uniformly distributed load of 500 lb. per linear ft. including its own weight. Compute M_2 and M_3, R_1, R_2, and R_3, M_A and M_B.

Ans. $M_2 = -5358$ ft.-lb.; $M_3 = -3571$ ft.-lb.; $R_1 = 1964$ lb.; $R_2 = 5715$ lb.; $R_3 = 4642$ lb.; $M_A = +3857$ ft.-lb.; $M_B = +1819$ ft.-lb.

4. A continuous girder 60 ft. long with three equal spans of 20 ft. each is to carry a superimposed load of 360 lb. per linear ft. With an allowable value of $S = 18{,}000$ lb./in.^{2}, select the lightest American Standard I-beam which will carry the load. Compute the reactions.

Ans. 7-in. 15.3-lb. I-beam; $R_1 = 3002$ lb.; $R_2 = 8257$ lb.

5. Compute the reactions of the shaft described in Problem 6, Art. 78.

Ans. $R_1 = 636$ lb.; $R_2 = 284$ lb.; $R_3 = 462$ lb.; $R_4 = -12$ lb.

80. Continuous Beam with Fixed Ends, and with Unequal Loads and Spans.—If a continuous beam is fixed at both ends and has unequal spans or unequal loads or both, each span is not equivalent to a beam fixed at both ends as was the case when the spans and loads were equal. If, at each end of the beam, there is assumed to be an auxiliary span, the length of which is zero, the application of the general theorem of three moments will give enough equations to solve for the moments at the fixed ends and at the supports. The shear at the fixed end at the left may then be obtained by using the left span as the free body and writing the moment equation with respect to the right end of the free body. The reaction at the first intermediate support is obtained by using the two spans at the left as the free body, with the shear at the fixed end now known. The equation of moments with respect to the right end of the free body gives the value of the reaction. This method is continued, adding the next span to the free body, until the entire beam is used. The shear at the fixed end at the right is then obtained by using the last span on the right as the free body. The results obtained may be checked by summing vertical forces on the entire beam as the free body.

EXAMPLE

Figure 127 represents a beam which is fixed at sections 2 and 5, and which is supported and loaded as shown. Compute the moments at the fixed ends and at the supports. Compute the shears V_2 and V_5, and the reactions R_3 and R_4.

Solution.—At the left of section 2 and at the right of section 5 auxiliary spans of length $l = 0$ are assumed. The general theorem of three moments is then written for points 1, 2, and 3, then for points 2, 3, and 4, and so on. The general equation is

$$M_Al_1 + 2M_B(l_1 + l_2) + M_Cl_2 = -\frac{w_1l_1^3}{4} - \frac{w_2l_2^3}{4}.$$

For points 1, 2, and 3,

$$0 + 40M_2 + 20M_3 = 0 - 600{,}000.$$

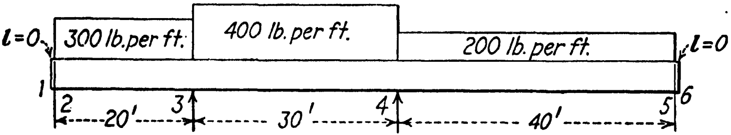


FIG. 127.

For points 2, 3, and 4,

$$20M_2 + 100M_3 + 30M_4 = -3{,}300{,}000.$$

For points 3, 4, and 5,

$$30M_3 + 140M_4 + 40M_5 = -5{,}900{,}000.$$

For points 4, 5, and 6,

$$40M_4 + 80M_5 + 0 = -3{,}200{,}000 - 0.$$

Solution of these four equations gives the following values:
$M_2 = -3333$ ft.-lb.; $M_3 = -23{,}333$ ft.-lb.; $M_4 = -30{,}000$ ft.-lb.; $M_5 = -25{,}000$ ft.-lb.

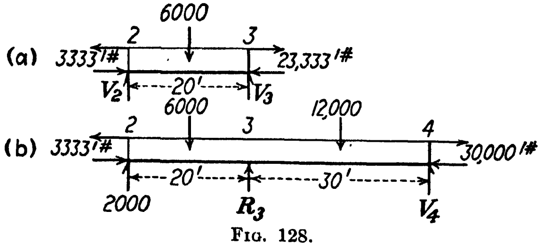


FIG. 128.

The span from section 2 to reaction 3 is shown as a free body in Fig. 128(a). The equation of moments with respect to a point on V_3 gives the following equation:

$$20V_2 + 23{,}333 = 3333 + 60{,}000.$$
$$V_2 = 2000 \text{ lb.}$$

The two spans from section 2 to reaction 4 are shown as a free body in Fig. 128(b). The equation of moments with respect to a point on V_4 gives the following equation:

$$30R_3 + 100{,}000 + 30{,}000 = 3333 + 180{,}000 + 240{,}000.$$
$$R_3 = 9778 \text{ lb.}$$

With the entire beam from section 2 to section 5 as the free body, (the student should draw the free-body diagram), the equation of moments with respect to V_5 gives the following equation:

$$40R_4 + 684{,}460 + 180{,}000 + 25{,}000 = 3333 + 160{,}000 + 660{,}000 + 480{,}000.$$
$$R_4 = 10{,}347 \text{ lb.}$$

With the span from reaction 4 to section 5 as the free body, the equation of moments with respect to the left end gives the following equation:

$$40V_5 + 30{,}000 = 160{,}000 + 25{,}000.$$
$$V_5 = 3875 \text{ lb.}$$

As a check, the sum of the vertical shears at the ends and the intermediate reactions is equal to the sum of the loads:

$$2000 + 9778 + 10{,}347 + 3875 = 6000 + 12{,}000 + 8000.$$

PROBLEMS

1. On the beam shown in Fig. 127, let $w_1 = 400$ lb./ft., $w_2 = 200$ lb./ft., and $w_3 = 300$ lb./ft. Solve for the moments at the ends and at the supports, the end shears, and the intermediate reactions.

Ans. $M_2 = -15{,}076$ ft.-lb.; $M_3 = -9848$ ft.-lb.; $M_4 = -28{,}788$ ft.-lb.; $M_5 = -45{,}606$ ft.-lb.; $V_2 = 4261$ lb.; $R_3 = 6108$ lb.; $R_4 = 9211$ lb.; $V_5 = 6420$ lb.

2. A beam 70 ft. long, fixed at both ends, has end spans of 30 ft. each and a middle span of 10 ft. The left span is loaded with 400 lb. per linear ft., the middle span with 500 lb. per linear ft., and the right span with 800 lb. per linear ft. Solve for the moments at the ends and at the supports, the end shears, and the intermediate reactions.

Ans. $M_2 = -36{,}803$ ft.-lb.; $M_3 = -16{,}394$ ft.-lb.; $M_4 = -40{,}939$ ft.-lb.; $M_5 = -69{,}530$ ft.-lb.; $V_2 = 6681$ lb.; $R_3 = 5362$ lb.; $R_4 = 16{,}002$ lb.; $V_5 = 12{,}953$ lb.

81. Theorem of Three Moments, Loads Concentrated.—

Figure 129 represents two adjoining spans of a continuous beam

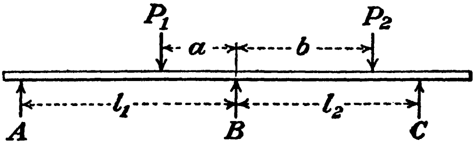


FIG. 129.

with lengths l_1 and l_2. Span l_1 carries a load P_1 at a distance of a from the right end and span l_2 carries a load P_2 at a distance of b from the left end. Consider first the second span, between reactions B and C. There are two expressions for the moment in this span.

FROM REACTION B TO LOAD P_2

$$EI\frac{d^2y}{dx^2} = M_B + V_Bx.$$

$$EI\frac{dy}{dx} = M_Bx + \frac{V_Bx^2}{2} + C_1.$$

FROM LOAD P_2 TO REACTION C

$$EI\frac{d^2y}{dx^2} = M_B + V_Bx - P_2(x - b).$$

$$EI\frac{dy}{dx} = M_Bx + \frac{V_Bx^2}{2} - \frac{P_2(x - b)^2}{2} + C_2.$$

At reaction B, $EI\frac{dy}{dx} = C_1$.

If $x = b$,

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

$$M_Bb + \frac{V_Bb^2}{2} + C_1 = M_Bb + \frac{V_Bb^2}{2} - \frac{P_2}{2}(b - b)^2 + C_2.$$

$$C_1 = C_2.$$

$$EIy = \frac{M_Bx^2}{2} + \frac{V_Bx^3}{6} + C_1x + C_3.$$

$$EIy = \frac{M_Bx^2}{2} + \frac{V_Bx^3}{6} - \frac{P_2}{6}(x - b)^3 + C_1x + C_4.$$

At B, $y = 0$ and $x = 0$, so $C_3 = 0$.

If $x = b$,

$$y = y.$$

$$\frac{M_Bb^2}{2} + \frac{V_Bb^3}{6} + C_1b = \frac{M_Bb^2}{2} + \frac{V_Bb^3}{6} - \frac{P_2}{6}(b - b)^3 + C_1b + C_4.$$

$$C_4 = 0.$$

At C, $y = 0$ and $x = l_2$.

$$0 = \frac{M_Bl_2^2}{2} + \frac{V_Bl_2^3}{6} - \frac{P_2}{6}(l_2 - b) + C_1l_2.$$

Also, at reaction C, from the original equation,

$$M_C = M_B + V_Bl_2 - P_2(l_2 - b).$$

By eliminating V_B from the last two equations,

$$0 = 2M_Bl_2 + M_Cl_2 + P_2l_2(l_2 - b) - \frac{P_2}{l_2}(l_2 - b)^3 + 6C_1.$$

If the origin is taken at reaction B and x is positive to the left, a corresponding equation for the first span will be obtained.

$$0 = 2M_Bl_1 + M_Al_1 + P_1l_1(l_1 - a) - \frac{P_1}{l_1}(l_1 - a)^3 + 6C_1'.$$

At reaction B, with x positive to the left,

$$C_1' = EI\frac{dy}{dx}.$$

The quantity C_1' is therefore equal numerically to C_1, but is opposite in sign, so

$$C_1' = -C_1.$$

By adding the last two equations,

$$M_Al_1 + 2M_B(l_1 + l_2) + M_Cl_2 = \frac{P_1}{l_1}(l_1 - a)^3 + \frac{P_2}{l_2}(l_2 - b)^3 - P_1l_1(l_1 - a) - P_2l_2(l_2 - b).$$

This is the general theorem of three moments for concentrated loads. If there are other loads, P_1', etc., P_2', etc., distant a', etc., b', etc., from point B, corresponding terms must be added to the general theorem.

If the loads and spans are equal and each load is in the middle of its span, the theorem of three moments for concentrated loads becomes

$$M_A + 4M_B + M_C = -\tfrac{3}{4}Pl.$$

If the ends of the beam are free or overhanging, the application of the theorem of three moments gives two less equations than there are supports. If the beam ends at a support, the moment there is zero, while if it overhangs and carries a load on the overhanging end, the moment at the end support is computed from static conditions. These two equations give the number necessary for solution. After the moments at the supports have been determined, the reactions are computed in the same manner as in Art. 79.

If the ends of the beam are fixed and the spans and loads are equal, with each load in the middle of its span, each span is under the same condition of loading and support as the beam fixed at both ends with a load in the middle of the span.

If the ends of a continuous beam are fixed and the spans are not equal, or if the loads are not equal, or are not placed in the middle of the spans, it is necessary to assume an auxiliary span of length $l = 0$ at each fixed end in order to obtain enough equations to make the solution. The end shears, the intermediate reactions, and the moment at each load may then be computed. The moment at one of the loads may be the largest and therefore the controlling moment.

EXAMPLE 1

Figure 130(*a*) shows a continuous beam with four concentrated loads. Compute the moment and the reaction at each support. Draw the shear diagram and compute the moment at each load

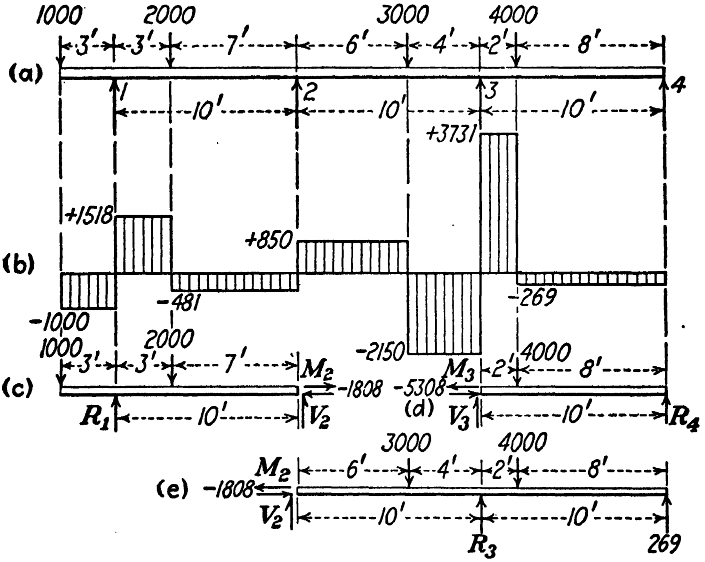


FIG. 130.

Solution.—Moment $M_1 = -3000$ ft.-lb.
Moment $M_4 = 0$.

The application of the general theorem of three moments gives two equations. For the first and second spans,

$$-30{,}000 + 40M_2 + 10M_3 = 200 \times 27 + 300 \times 64 - 20{,}000 \times 3 - 30{,}000 \times 4.$$

For the second and third spans,

$$10M_2 + 40M_3 = 300 \times 216 + 400 \times 512 - 30{,}000 \times 6 - 40{,}000 \times 8.$$

$$M_2 = -1808 \text{ ft.-lb.}$$
$$M_3 = -5308 \text{ ft.-lb.}$$

With the left span and the overhanging part as the free body, Fig. 130(*c*),

$$10R_1 + 1808 = 13{,}000 + 14{,}000.$$
$$R_1 = 2519 \text{ lb.}$$

With the third span as the free body, Fig. 130(*d*),

$$10R_4 + 5308 = 8000.$$
$$R_4 = 269 \text{ lb.}$$

With the second and third spans together as the free body, Fig. 130(*e*),

$$10R_3 + 5380 + 1808 = 18{,}000 + 48{,}000.$$
$$R_3 = 5881 \text{ lb.}$$

By summing vertical forces on the entire beam,

$$R_2 = 1331 \text{ lb.}$$

Figure 130(*b*) shows the shear diagram. The moment under the 2000-lb. load is $M_1' = -1000 \times 3 + 1519 \times 3 = +1557$ ft.-lb. The moment under the 3000-lb. load is $M_2' = +3290$ ft.-lb. The moment under the 4000-lb. load is $M_3' = +2152$ ft.-lb.

EXAMPLE 2

A beam 60 ft. long is supported at the ends and at points 10 ft. and 30 ft. from the left end. It is loaded with a concentrated load of 4000 lb. at a point 4 ft. from the left end, one of 3000 lb. at a point 22 ft. from the left end, one of 1500 lb. at a point 20 ft. from the right end, and one of 6000 lb. at a point 6 ft. from the right end. Compute the moments at the second and third supports.

Solution.—Moments M_1 and $M_4 = 0$.
The general theorem of three moments gives two equations. For the first and second spans,

$$60M_2 + 20M_3 = 400 \times 64 + 150 \times 512 - 40{,}000 \times 4 - 60{,}000 \times 8.$$

For the second and third spans,

$$20M_2 + 100M_3 = 150 \times 1728 + 50 \times 8000 + 200 \times 216 - 60{,}000 \times 12 - 45{,}000 \times 20 - 180{,}000 \times 6.$$
$$M_2 = -2465 \text{ ft.-lb.}$$
$$M_3 = -19{,}483 \text{ ft.-lb.}$$

EXAMPLE 3

Figure 131 shows a beam fixed at both ends, supported at two intermediate points, and loaded with three concentrated loads. Compute the moments at the fixed ends, at the supports, and under the loads. Compute the shear at each end and the intermediate reactions.

Solution.—As in Art. 80, an auxiliary span of length $l = 0$ is assumed at each end. For the spans 0-1 and 1-2, the general equation of three moments gives the following equation:

$$40M_1 + 20M_2 = 100 \times 512 - 40{,}000 \times 8.$$

For spans 1-2 and 2-3,

$$20M_1 + 60M_2 + 10M_3 = 100 \times 1728 + 300 \times 125 - 40{,}000 \times 12 - 30{,}000 \times 5.$$

For spans 2-3 and 3-4,

$$10M_2 + 52M_3 + 16M_4 = 300 \times 125 + 100 \times 216 - 30{,}000 \times 5 - 25{,}600 \times 6.$$

For spans 3-4 and 4-5,

$$16M_3 + 32M_4 = 100 \times 1000 - 25{,}600 \times 10.$$

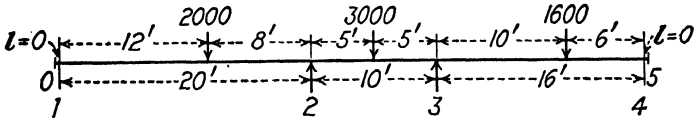


FIG. 131.

The solution of these four equations gives the following values:

$$\begin{aligned} M_1 &= -4127 \text{ ft.-lb.} \\ M_2 &= -5185 \text{ ft.-lb.} \\ M_3 &= -2606 \text{ ft.-lb.} \\ M_4 &= -3572 \text{ ft.-lb.} \end{aligned}$$

With span 1-2 as the free body, the equation of moments with respect to an axis at R_2 is

$$\begin{aligned} 20V_1 + 5185 &= 4127 + 16{,}000. \\ V_1 &= 747 \text{ lb.} \end{aligned}$$

With the part of the beam from section 1 to the 2000-lb. load as the free body, the equation of moments with respect to an axis at the load is

$$M_1' = 747 \times 12 - 4127 = +4837 \text{ ft.-lb.}$$

With spans 1-2 and 2-3 together as the free body,

$$\begin{aligned} 10R_2 + 30 \times 747 + 2606 &= 4127 + 2000 \times 18 + 3000 \times 5. \\ R_2 &= 3011 \text{ lb.} \\ M_2' &= 747 \times 25 + 3011 \times 5 - 4127 - 200 \times 13. \\ M_2' &= +3603 \text{ ft.-lb.} \end{aligned}$$

With span 3-4 as the free body,

$$\begin{aligned} 16V_4 + 2606 &= 1600 \times 10 + 3572. \\ V_4 &= 1060 \text{ lb.} \end{aligned}$$

With spans 2-3 and 3-4 together as the free body,

$$\begin{aligned} 10R_3 + 26 \times 1060 + 5185 &= 15{,}000 + 32{,}000 + 3572. \\ R_3 &= 1783 \text{ lb.} \\ M_3' &= 1060 \times 6 - 3572. \\ M_3' &= 2788 \text{ ft.-lb.} \end{aligned}$$

The results just obtained for the end shears and the reactions may be checked by summing the vertical forces on the entire beam as the free body.

PROBLEMS

1. A beam 30 ft. long has three equal spans of 10 ft. each, and carries a load of 1600 lb. at the middle of each span. Compute the moment and the reaction at each support and the moment under each load.

Ans. $M_2 = M_3 = -2400$ ft.-lb.; $R_1 = 560$ lb.; $R_2 = 1840$ lb.; $M_1' = M_3' = +2800$ ft.-lb.; $M_2' = +1600$ ft.-lb.

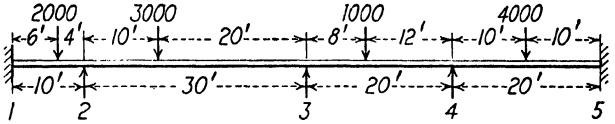


FIG. 132.

2. Solve Problem 1 if only the center span is loaded.

Ans. $M_2 = M_3 = -1200$ ft.-lb.; $R_1 = R_4 = -120$ lb.; $R_2 = R_3 = 920$ lb.; $M_2' = +2800$ ft.-lb.

3. Solve Problem 1 if only the end spans are loaded.

Ans. $M_2 = M_3 = -1200$ ft.-lb.; $R_1 = R_4 = 680$ lb.; $R_2 = R_3 = 920$ lb.; $M_1' = M_3' = +3400$ ft.-lb.

4. Solve Problem 1 if only the first span is loaded.

Ans. $M_2 = -1600$ ft.-lb.; $M_3 = +400$ ft.-lb.; $R_1 = 640$ lb.; $R_2 = 1160$ lb.; $R_3 = -240$ lb.; $R_4 = 40$ lb.; $M_1' = +3200$ ft.-lb.

5. A beam 80 ft. long is fixed at both ends and is supported and loaded as shown in Fig. 132. Compute all of the moments, the shear at each end, and the intermediate reactions due to the concentrated loads.

Ans. $M_1 = +2593$ ft.-lb.; $M_2 = -11{,}906$ ft.-lb.; $M_3 = -5008$ ft.-lb.; $M_4 = -4775$ ft.-lb.; $M_5 = -12{,}612$ ft.-lb.; $M_1' = -1307$ ft.-lb.; $M_2' = +10{,}393$ ft.-lb.; $M_3' = -116$ ft.-lb.; $M_4' = +11{,}308$ ft.-lb.; $V_1 = -650$ lb.; $R_2 = 4880$ lb.; $R_3 = 1382$ lb.; $R_4 = 1996$ lb.; $V_5 = 2392$ lb.

CHAPTER IX

BEAMS OF TWO MATERIALS

82. Wood Beam with Steel Plates on the Sides.—In Art. 7 it was shown that if a bar which is subjected to tension or compression is composed of two materials for which the values of E are different, the same unit deformation δ will be produced by unit stresses which are proportional to the moduli of elasticity of the two materials.

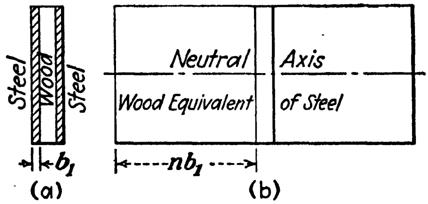


FIG. 133.

$$\delta = \frac{S_1}{E_1} = \frac{S_2}{E_2}.$$

The same principle holds true in the case of a beam composed of two different materials. Figure 133(a) shows the cross section of a wooden beam which is reinforced with steel plates on the sides. If n is the ratio of the modulus of elasticity of the steel to that of the wood, the unit stress at any point in the steel is n times the unit stress in the wood at the same distance from the neutral axis. The steel is equivalent, therefore, to n times as much area of wood, as shown in Fig. 133(b). The moment of inertia of the equivalent section including the original cross section of the wooden beam is then computed. The equation $S = Mc/I$ gives the unit stress in the wood and in the wood equivalent of the steel. The actual unit stress in the steel is n times as great.

EXAMPLE

A simple wooden beam 8 in. wide, 10 in. deep, and 30 ft. long is to be reinforced with a steel plate 10 in. deep on each side. Allowable S_w = 1250 lb./in.^{2}; S_s = 18,000 lb./in.^{2}; n = 16. Compute the thickness of the steel plates required if the beam is to support a load of 6000 lb. at the middle.

Solution.—Let b be the total width of the equivalent section required. The depth is 10 in. and the section modulus is $I/c = b \times 100/6$. The stress in the steel governs, since $nS_w = 16 \times 1250 = 20{,}000$ lb./in.^{2}, which is higher than the allowable value. When the unit stress in the steel is 18,000 lb./in.^{2}, the unit stress in the wood equivalent of the steel is 18,000/16 = 1125 lb./in.^{2}. The equation $M/S = I/c$ gives

$$6000 \times \frac{90}{1125} = b \times \frac{100}{6}.$$

$$b = 28.8 \text{ in.}$$

$$2nb_1 = 28.8 - 8 = 20.8 \text{ in.}$$

$$b_1 = \frac{20.8}{32} = 0.65 \text{ in.}$$

The necessary commercial thickness of the plates would be $^{11}\!/_{16}$ in.

PROBLEMS

1. Solve the example above if the depth of the steel plates is only 8 in. and the plates are placed symmetrically on the beam.

Ans. 1.092 in. (Use 1⅛-in. plates.)

2. A wooden beam 6 in. wide, 8 in. deep, and 25 ft. long has a steel plate ½ in. wide and 8 in. deep on each side. E_w = 1,500,000 lb./in.^{2}; E_s = 30,000,000 lb./in.^{2}. Compute the maximum unit stress in each material due to a total uniformly distributed load of 5000 lb.

Ans. S_w = 676 lb./in^2.; S_s = 13,520 lb./in.^{2}.

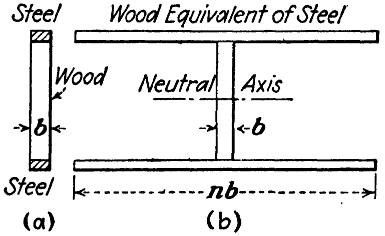


FIG. 134.

83. Wood Beam with Steel Plates on Top and Bottom.—

Figure 134(a) represents the cross section of a wooden beam reinforced with steel plates at the top and bottom. If n is the ratio of the modulus of elasticity of the steel to that of the wood, the stress at any point in the steel is n times what it would be if there were wood at that point. The steel is therefore equivalent to n times as much area of wood, as shown in Fig. 134(*b*), and the beam is equivalent to an I-section, all composed of wood.

The moment of inertia of the equivalent I-section of wood is computed and the expression $S = My/I$ gives the maximum unit stress in the wood, y being the distance from the neutral axis to the outer fiber of the actual wood. The expression $S = Mc/I$, in which c is the distance from the neutral axis to the outer fiber of the wood equivalent, gives the maximum unit fiber stress in the wood equivalent. The actual unit stress in the steel is n times this stress.

EXAMPLE

A wooden cantilever beam 3 in. wide, 6 in. deep, and 4 ft. long is reinforced with a plate of steel 3 in. wide and $\frac{3}{16}$ in. thick on the top and bottom. Using the value of n as 20, compute the maximum unit stress in each material due to a load of 1000 lb. at the free end.

Solution.—The width of the wood equivalent of the steel is $nb = 60$ in. The moment of inertia of the actual wood area is

$$I_w = 3 \times \frac{6^3}{12} = 54 \text{ in.}^4.$$

The moment of inertia of the two wood equivalent areas is

$$\begin{aligned} I_e &= 2[\tfrac{1}{12}60 \times (\tfrac{3}{16})^3 + 11.25 \times 3.094^2] \\ &= 215.42 \text{ in.}^4. \\ \text{Total } I &= 54 + 215.42 = 269.42 \text{ in.}^4. \\ M &= 48{,}000 \text{ in.-lb.} \\ S_w &= \frac{48{,}000 \times 3}{269.42} = 535 \text{ lb./in.}^2. \\ S_s &= \frac{48{,}000 \times 3.1875 \times 20}{269.42} = 11{,}360 \text{ lb./in.}^2. \end{aligned}$$

PROBLEMS

1. A simple wooden beam 6 in. wide, 18 in. deep, and 20 ft. long has a plate of steel 6 in. wide and ½ in. thick on the top and bottom. $E_w = 1{,}750{,}000$ lb./in.²; $E_s = 29{,}000{,}000$ lb./in.². Allowable $S_w = 1300$ lb./in.²; $S_s = 18{,}000$ lb./in.². Compute the safe load at the middle of the beam.
Ans. 21,800 lb.

2. In Problem 1, how thick must the plates be made if the load to be carried is 30,000 lb.? *Ans.* 0.76 in.

3. A cantilever beam 10 ft. long is composed of a wooden beam 5 in. wide and 8 in. deep reinforced on the top and bottom with plates of steel 4 in. wide and ¼ in. thick. Using $n = 18$, compute the maximum stress in each material due to a load of 1600 lb. at the free end.
Ans. $S_w = 930$ lb./in.²; $S_s = 17{,}800$ lb./in.².

84. Wood Beam with Steel Plate on the Tension Face Only.

Figure 135(*a*) represents the cross section of a simple wooden beam with a steel plate on the bottom only. Figure 135(*b*) represents the cross section with the steel replaced by its wood equivalent. The section is a T-section, for which it is necessary to compute the location of the neutral axis. The maximum unit stress in the wood is at the upper part of the cross section.

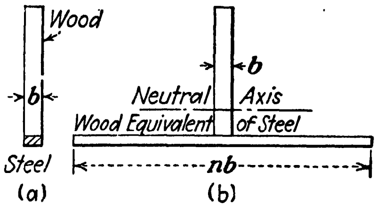


FIG. 135.

EXAMPLE

A simple wooden beam 4 in. wide, 8 in. deep, and 10 ft. long has a strip of steel 4 in. wide and ¼ in. thick, fastened to the bottom as shown in Fig. 135. If the modulus of elasticity of the wood is 1,500,000 lb./in.^{2} and that of the steel is 30,000,000 lb./in.^{2}, what are the maximum unit stresses caused by a load of 2400 lb. at the middle?

Solution:

$$n = \frac{30{,}000{,}000}{1{,}500{,}000} = 20.$$

The width of the wood equivalent of the steel is nb, or 80 in., and its area is $A_1 = 80 \times \frac{1}{4} = 20$ sq. in. The area of the actual wood is $A_2 = 32$ sq. in. The distance of the neutral axis from the bottom of the steel is given by the equation

$$\bar{y} = \frac{20 \times 0.125 + 32 \times 4.25}{52} = 2.66 \text{ in.}$$

The moment of inertia of the equivalent area with respect to the axis at the bottom of the actual wood is

$$I = \frac{4 \times 512}{3} + \frac{80}{3 \times 64} = 683.1 \text{ in.}^4.$$

The moment of inertia of this area with respect to the neutral axis is

$$I_o = 683.1 - 52 \times 2.41^2 = 381.1 \text{ in.}^4.$$

The maximum unit stress in the wood is

$$S_w = \frac{2400 \times 120 \times 5.59}{4 \times 381.1} = 1055 \text{ lb./in.}^2.$$

The maximum unit stress in the steel is

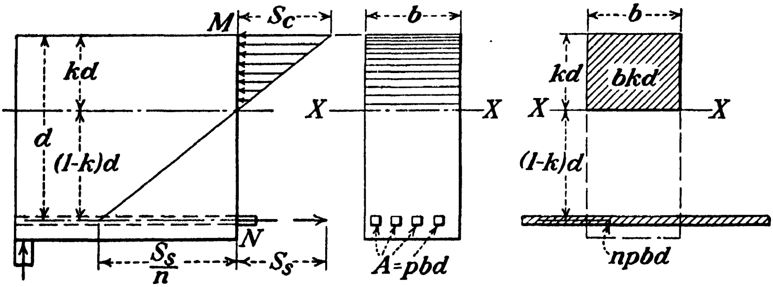
$$S_s = \frac{2400 \times 120 \times 2.66 \times 20}{4 \times 381.1} = 10{,}070 \text{ lb./in.}^2.$$

PROBLEMS

1. A simple wooden beam 10 in. wide, 24 in. deep, and 30 ft. long has a plate of steel 10 in. wide and 1 in. deep on the bottom. Allowable S_w = 1200 lb./in.^{2}; S_s = 16,000 lb./in.^{2}. Using n = 20, compute the total safe uniformly distributed load. *Ans.* 43,000 lb.

2. Solve Problem 1 if the steel plate is only ½ in. thick. *Ans.* 37,800 lb.

85. Analysis of Reinforced Concrete Beam.—The concrete and steel beam, commonly called the *reinforced concrete* beam, presents a slightly different problem from that of the wood and steel beam. The steel is placed on the tension side of the beam and is assumed to carry all of the tension. At loads considerably less than the allowable load, hair cracks will develop in the beam on the tensile side, showing that the concrete has failed in tension and ceases to carry any stress except for a small area of the cross section near the neutral axis.



(a) (b) (c)

FIG. 136.

Figure 136 shows the notation commonly used for reinforced concrete beams. Figure 136(*a*) represents a side view of a beam supported at the ends. *MN* is a cross section at any point along the beam, and in the figure the internal forces are shown acting on the beam at this section. Figure 136(*b*) shows a view of the cross section, in which *b* is the width, *d* is the depth from the top of the beam to the middle of the steel (the area of concrete below the middle of the steel being neglected), *p* is the percentage of steel reinforcement, that is, the ratio of the area *A* of the steel to the area *bd* of the concrete, *kd* is the distance from the top to the neutral axis of the cross section, *n* is the ratio of the modulus of elasticity of the steel to that of the concrete, S_c is the unit stress in the concrete at the outer fiber, and S_s is the unit stress in the steel.

Figure 136(c) shows the cross section of the beam, with the effective area of the concrete and the concrete equivalent of the steel shown shaded. In order that axis X-X shall be the neutral axis, it is necessary that the moment of the area bkd with respect to axis X-X shall be the same as the moment of the area $npbd$.

$$bkd \times \frac{kd}{2} = npbd \times d(1 - k).$$

$$k^2 + 2pnk - 2pn = 0.$$

$$k = \sqrt{2pn + (pn)^2} - pn.$$

With the value of k known, the moment of inertia of the equivalent section is given by the expression,

$$I = \frac{bk^3d^3}{3} + npbd^3(1 - k)^2.$$

The unit stresses are given by the equations,

$$S_c = \frac{Mkd}{I},$$

and

$$S_s = \frac{nM(1 - k)d}{I}.$$

EXAMPLE 1

A reinforced concrete beam is 20 ft. long, 10 in. wide, and 17 in. deep, with five bars, each $\frac{5}{8}$ in. in diameter, placed with their centers 1 in. above the bottom of the beam. Using the value of n as 15, compute the unit stress in the concrete and in the steel due to a load of 350 lb. per linear ft. in addition to its own weight.

Solution:

$$A = 5 \times 0.3068 = 1.534 \text{ sq. in.}$$

$$d = 16 \text{ in.}$$

$$p = \frac{A}{bd} = \frac{1.534}{160} = 0.0096.$$

$$pn = 0.144.$$

$$k = \sqrt{2pn + (pn)^2} - pn = 0.412.$$

$$kd = 6.59 \text{ in.}$$

$$(1 - k)d = 9.41 \text{ in.}$$

$$I = \frac{10 \times 6.59^3}{3} + 15 \times 1.534 \times 9.41^2 = 2991 \text{ in.}^4$$

$$W = \frac{20 \times 10 \times 17 \times 150}{144} + 20 \times 350 = 10{,}540 \text{ lb.}$$

$$M = \frac{Wl}{8} = \frac{10{,}540 \times 240}{8} = 316{,}200 \text{ in.-lb.}$$

$$S_c = \frac{316{,}200 \times 6.59}{2991} = 696 \text{ lb./in.}^2.$$

$$S_s = \frac{15 \times 316{,}200 \times 9.41}{2991} = 14{,}920 \text{ lb./in.}^2.$$

EXAMPLE 2

A continuous reinforced concrete beam with fixed ends is 16 in. wide, 18 in. deep, and has 16-ft. spans. The reinforcement consists of four steel bars each ¾ in. square, placed with their centers 1 in. above the bottom where the moment is positive and 1 in. below the top where the moment is negative. If n is 15, allowable S_c is 650 lb./in.², and allowable S_s is 16,000 lb./in.², what superimposed load per linear foot may be placed upon the beam?

Solution:

$$A = 4 \times \tfrac{3}{4} \times \tfrac{3}{4} = 2.25 \text{ sq. in.}$$

$$d = 17 \text{ in.}$$

$$p = \frac{A}{bd} = \frac{2.25}{16 \times 17} = 0.00826.$$

$$pn = 0.1239.$$

$$k = 0.389.$$

$$kd = 6.61 \text{ in.}$$

$$(1 - k)d = 10.39 \text{ in.}$$

$$I = \frac{16 \times 6.61^3}{3} + 15 \times 2.25 \times 10.39^2 = 5184 \text{ in.}^4.$$

When the unit stress in the concrete at the outer fiber is 650 lb./in.², the unit stress in the steel is

$$S_s = \frac{650 \times 15 \times 10.39}{6.61} = 15{,}340 \text{ lb./in.}^2.$$

Since 16,000 lb./in.² is the allowable stress in the steel, the unit stress in the concrete governs.

As shown in Art. 78, the maximum moment in this continuous beam is

$$M = \frac{Wl}{12}.$$

$$S_c = \frac{Mkd}{I}.$$

$$650 = \frac{W \times 192 \times 6.61}{12 \times 5184}.$$

$$W = 31{,}800 \text{ lb.}$$

The weight of the beam itself is

$$W_1 = \frac{16 \times 18 \times 16 \times 150}{12 \times 12} = 4800 \text{ lb.}$$

The total superimposed load on the beam may be

$$W_2 = 31{,}800 - 4800 = 27{,}000 \text{ lb.}$$

$$\frac{27{,}000}{16} = 1688 \text{ lb. per linear ft.}$$

PROBLEMS

1. Using $n = 15$, compute the value of k for 0.85 per cent reinforcement; for 0.80 per cent. *Ans.* 0.393; 0.384.

2. A simple concrete beam 20 ft. long is 6 in. wide, 12 in. deep, and has four bars of steel each ½ in. in diameter placed 1 in. above the bottom. Compute the maximum stress in each material caused by a concentrated load of 2000 lb. at the middle of the beam. Use $n = 15$.

Ans. $S_c = 870$ lb./in.²; $S_s = 16{,}350$ lb./in.².

3. A reinforced concrete beam 15 in. wide, 24 in. deep, and 36 ft. long is supported at the ends and at the middle. It is reinforced with six ¾-in. round rods placed 1.5 in. from the tension face. Using $E_s = 29{,}000{,}000$ lb./in.², $E_c = 2{,}200{,}000$ lb./in.², allowable $S_c = 800$ lb./in.², and allowable $S_s = 18{,}000$ lb./in.², compute the total superimposed load which may be placed on the beam. Concrete weighs 150 lb./cu. ft. *Ans.* 56,500 lb.

86. Design of Reinforced Concrete Beams.—In the design of reinforced concrete beams, the value of p, the percentage of reinforcement, must be assumed. With p and n known, the value of k can be computed. With given allowable values of the unit stresses S_c and S_s, the one which governs in the design may be determined as follows:

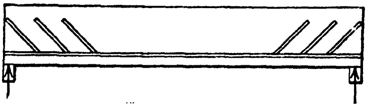


FIG. 137.

When the stress in the concrete has reached the value of S_c, the stress in the steel is

$$S_s = \frac{nS_c(1 - k)}{k}.$$

If this value of S_s is smaller than the allowable value of S_s, the stress in the concrete S_c governs, while if it is larger, the stress in the steel governs.

If the stress in the concrete governs, the equation $S_c = Mkd/I$ is solved for bd^2.

If the stress in the steel governs, the equation $S_s = \dfrac{nM(1 - k)d}{I}$ is solved for bd^2. Since both b and d occur in the expression, it is necessary to assume one or the other, or to assume some relation between them, such as $d = b$ or $d \doteq 2b$.

It was shown in Art. 16 that the resultant of the direct vertical shearing stress and the induced horizontal shearing stress was a diagonal tensile stress of equal unit intensity. One method of providing for this diagonal tensile stress is to bend up some of the reinforcing bars near the ends of the beam as shown in Fig. 137. Another method is to place vertical reinforcing bars, usually in the form of stirrups, to carry the vertical shear, as shown in Fig. 138.

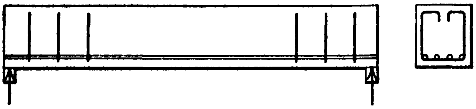


FIG. 138.

EXAMPLE 1

Using allowable S_c as 650 lb./in.², allowable S_s as 16,000 lb./in.², n as 15, and p as 0.008, design a simple reinforced concrete beam for a span of 12 ft. to carry a uniformly distributed load of 1000 lb. per linear ft. including its own weight. Assume $b = d$.

Solution.—From Problem 1, Art. 85, $k = 0.384$. $(1 - k) = 0.616$.

When the stress in the concrete reaches its allowable value of $S_c = 650$ lb./in.², the value of the unit stress in the steel is

$$S_s = \frac{nS_c(1-k)}{k} = \frac{15 \times 650 \times 0.616}{0.384} = 15{,}600 \text{ lb./in.}^2.$$

The unit stress in the concrete governs, since this value of S_s is less than the allowable stress in the steel.

$$S_c = \frac{Mkd}{I}.$$

$$W = 12{,}000 \text{ lb.} \qquad M = \tfrac{1}{8}Wl = 216{,}000 \text{ in.-lb.}$$

$$I = \tfrac{1}{3}b \times 0.384^3d^3 + 0.12bd \times 0.616^2d^2.$$

Since $b = d$, $I = 0.0644d^4$.

$$650 = \frac{216{,}000 \times 0.384d}{0.0644d^4}.$$

$$d^3 = 1980.$$

$$d = b = 12.56 \text{ in.}$$

$$bd = 12.56 \times 12.56 = 158 \text{ sq. in.}$$

$$A = pbd = 0.008 \times 158 = 1.264 \text{ sq. in.}$$

Four $\frac{5}{8}$-in. round bars will give an area of 1.266 sq. in. If 1 in. of concrete is to extend below the center of the reinforcing bars, the beam will be 13.56 in. deep.

The weight of the beam itself will be

$$w = \frac{12.56 \times 13.56 \times 150}{144} = 178 \text{ lb. per lin. ft.}$$

The superimposed load can be only $1000 - 178 = 822$ lb. per lin. ft.

EXAMPLE 2

Using allowable S_c as 700 lb./in.², allowable S_s as 18,000 lb./in.², n as 15, and p as 0.007, design a continuous reinforced concrete beam with fixed ends and spans of 20 ft. to carry a uniformly distributed load of 1200 lb. per linear ft. including its own weight. Assume $d = 2b$.

Solution:

$$pn = 0.007 \times 15 = 0.105.$$
$$k = \sqrt{0.21 + 0.011} - 0.105 = 0.365.$$
$$(1 - k) = 0.635.$$

When the stress in the concrete reaches 700 lb./in.², the value of the unit stress in the steel is

$$S_s = \frac{15 \times 700 \times 0.635}{0.365} = 18{,}300 \text{ lb./in.}^2.$$

Since this value is greater than the allowable unit stress in the steel, the allowable stress of 18,000 lb./in.² governs the design.

$$S_s = \frac{nM(1 - k)d}{I}.$$
$$W = 24{,}000 \text{ lb.}$$
$$M = \frac{Wl}{12} = 480{,}000 \text{ in.-lb.}$$
$$I = \frac{0.365^3bd^3}{3} + 15 \times 0.007 \times 0.635^2bd^3.$$
$$I = 0.0586bd^3.$$
$$18{,}000 = \frac{15 \times 480{,}000 \times 0.635d}{0.0586bd^3}.$$
$$bd^2 = 4340.$$

Since $d = 2b$,

$$bd^2 = 4b^3.$$
$$b^3 = 1085.$$
$$b = 10.3 \text{ in.}$$
$$d = 20.6 \text{ in.}$$
$$bd = 10.3 \times 20.6 = 212 \text{ sq. in.}$$
$$A = pbd = 0.007 \times 212 = 1.484 \text{ sq. in.}$$

Six ½-in. square bars will give an area of 1.5 sq. in.; or five ⅝-in. round bars will give an area of 1.534 sq. in.

If the concrete extends 1.4 in. below the center of the reinforcing bars, the beam will be 22 in. deep. The weight of the beam itself will be

$$w = \frac{10.3 \times 22 \times 150}{144} = 236 \text{ lb. per lin. ft.}$$

The superimposed load can be only 1200 − 236 = 964 lb. per lin. ft.

PROBLEMS

1. Using allowable $S_c = 500$ lb./in.², allowable $S_s = 15{,}000$ lb./in.², $n = 15$, and $p = 0.0075$, design a simple reinforced concrete beam for a span of 20 ft. to carry a total uniformly distributed load of 8000 lb. including its own weight. Assume $d = 3b$.

Ans. $b = 6.88$ in.; $d = 20.64$ in.; $A = 1.068$ sq. in.

2. Using allowable $S_c = 750$ lb./in.², allowable $S_s = 16{,}000$ lb./in.², $n = 12$, and $p = 0.008$, design a reinforced concrete beam with fixed ends for a span of 15 ft. and a total uniformly distributed load of 20,000 lb. including its own weight. Assume $d = 1.5b$.

Ans. $b = 10.56$ in.; $d = 15.84$ in.; $A = 1.34$ sq. in.

87. Balanced Reinforcement.—It was shown in Art. 86 that when the stress in the concrete has reached the value S_c, the stress in the steel is

$$S_s = \frac{nS_c(1 - k)}{k}.$$

If this expression is solved for k,

$$k = \frac{nS_c}{nS_c + S_s}.$$

As shown in Art. 85, the value of k in terms of pn is

$$k = \sqrt{2pn + (pn)^2} - pn.$$

If k is eliminated between these two equations,

$$p = \frac{nS_c^2}{2S_s(nS_c + S_s)}.$$

This expression gives the percentage of steel necessary in order that the stress in the steel shall be S_s when the stress in the concrete is S_c. If the percentage of steel is greater than this amount, the concrete will reach its allowable stress first. If the percentage is less than this amount, the steel will reach its allowable stress first.

PROBLEMS

1. Compute the value of p for balanced reinforcement if $S_c = 600$ lb./in.^{2}, $S_s = 17{,}500$ lb./in.^{2}, and $n = 15$. *Ans.* 0.00582.

2. Using $S_c = 650$ lb./in.^{2}, $S_s = 18{,}000$ lb./in.^{2}, and $n = 15$, design a simple reinforced concrete beam 30 ft. long with balanced reinforcement to support a total uniformly distributed load of 30,000 lb. including its own weight. Assume $b = 2d$.

Ans. $b = 37.6$ in.; $d = 18.8$ in.; $A = 4.48$ sq. in.

CHAPTER X

BEAMS OF CONSTANT STRENGTH

88. Cantilever Beam of Constant Strength.—A beam of constant strength is one in which the unit flexural stress S in the outer fibers is the same at all sections. Since $M/S = I/c$, the section modulus I/c must vary as the moment M if S is to be constant. Since in general the moment in a beam varies with the length, this requires that the section modulus shall also vary with the length.

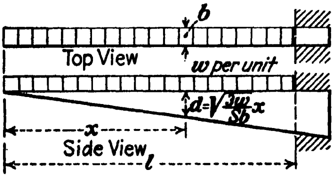


FIG. 139.

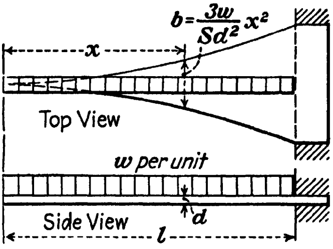


FIG. 140.

If the load is w per unit of length uniformly distributed, the moment at any section at a distance x from the free end is $M = wx^2/2$. (The negative sign may be disregarded.) For a beam with a rectangular cross section, the equation $M/S = I/c$ gives

$$\frac{wx^2}{2S} = \frac{bd^2}{6}.$$

If the width b is constant, as shown in Fig. 139, the variable depth d is given by the expression,

$$d = \sqrt{\frac{3w}{Sb}}x.$$

If the depth d is constant, as shown in Fig. 140, the variable width b is given by the expression,

$$b = \frac{3w}{Sd^2}x^2.$$

For a beam with a circular cross section,

$$\frac{wx^2}{2S} = \frac{\pi r^3}{4}.$$

$$r = \sqrt[3]{\frac{2wx^2}{\pi S}}.$$

If the load is a concentrated load P at the free end, the moment at any section at a distance x from the free end is Px. For a beam with a rectangular cross section, the equation $M/S = I/c$ gives

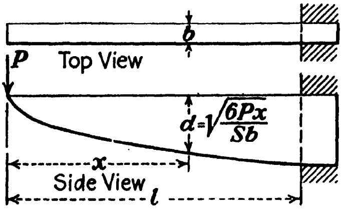


FIG. 141.

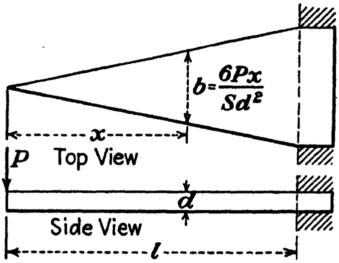


FIG. 142.

$$\frac{Px}{S} = \frac{bd^2}{6}.$$

If the width b is constant, as shown in Fig. 141, the variable depth d is given by the expression,

$$d = \sqrt{\frac{6Px}{Sb}}.$$

If the depth d is constant, as shown in Fig. 142, the variable width b is given by the expression,

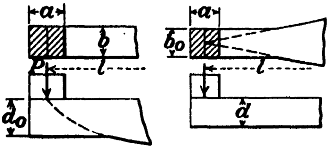
$$b = \frac{6Px}{Sd^2}.$$

For a beam with a circular cross section,

$$\frac{Px}{S} = \frac{\pi r^3}{4}.$$

$$r = \sqrt[3]{\frac{4Px}{\pi S}}.$$

It is evident that the load P cannot be applied on a point or along a line at the extreme end of the beam, as shown in Figs. 141 and 142, but modification of the end must be made both for bearing and shear. In some cases, also, the end must be modified for either the fixed or flexible attachment of the load P.



(a) (b)

FIG. 143.

Figure 143(a) shows the modification of the end of the beam shown in Fig. 141 for bearing and for shear. If S_c is the allowable unit bearing stress, the equation $P = S_c ab$ gives the distance a over which the load P must extend. The length of the beam must then be $l + a/2$.

If S_s is the allowable unit shearing stress (either vertical or horizontal), the average unit shearing stress is $\frac{2}{3}S_s$, as shown in Art. 54. The equation $P = \frac{2}{3}S_s bd_o$ gives the required depth d_o at the end.

In a similar manner, Fig. 143(b) shows the modification of the end of the beam shown in Fig. 142. The width b_o at the end necessary on account of the shear is given by the equation $P = \frac{2}{3}S_s b_o d$. The equation $P = S_c ab_o$ gives the distance a over which the load must extend.

EXAMPLE 1

A cantilever beam of constant strength and rectangular cross section is to have the depth twice the width. The beam is to be 4 ft. long and is to be loaded with a uniformly distributed load of 480 lb. per linear ft. Allowable $S = 3000$ lb./in.^{2}. Compute the dimensions at the middle and at the fixed end.

Solution.—Since $d = 2b$, $I/c = \frac{2}{3}b^3$. Unit weight $w = 40$ lb. per linear in. For $x = 24$,

$$\frac{2}{3}b^3 = \frac{40 \times 24^2}{2 \times 3000}.$$

$$b_{24} = 1.79 \text{ in.}; \qquad d_{24} = 3.58 \text{ in.}$$

For $x = 48$,

$$\frac{2}{3}b^3 = \frac{40 \times 48^2}{2 \times 3000}.$$

$$b_{48} = 2.85 \text{ in.}; \qquad d_{48} = 5.70 \text{ in.}$$

EXAMPLE 2

A cantilever beam of constant strength is 5 ft. long, 3 in. wide, and carries a concentrated load of 1200 lb. at the free end. The allowable unit stress in bending is 1000 lb./in.^{2}, that in bearing is 300 lb./in.^{2}, and that in shear is 120 lb./in.^{2}. Compute the depth at the free end and at each 1-ft. point to the fixed end. Compute also the extra length required for bearing.

Solution.—At the free end, the depth d_o is given by the equation,

$$d_o = \frac{3P}{2S_sb}.$$

$$d_o = \frac{3 \times 1200}{2 \times 120 \times 3} = 5 \text{ in.}$$

At a point 1 ft. from the free end, the moment is

$$M_1 = 14{,}400 \text{ in.-lb.}$$

$$d_1^2 = \frac{6 \times 14{,}400}{1000 \times 3} = 28.8.$$

$$d_1 = 5.36 \text{ in.}$$

At a point 2 ft. from the free end, the moment is

$$M_2 = 28{,}800 \text{ in.-lb.}$$

$$d_2^2 = 2 \times 28.8.$$

$$d_2 = 7.58 \text{ in.}$$

$$d_3^2 = 3 \times 28.8.$$

$$d_3 = 9.29 \text{ in.}$$

$$d_4 = 10.72 \text{ in.}$$

$$d_5 = 12 \text{ in.}$$

The extra length that the beam must extend beyond the middle of the load is given by the equation,

$$\frac{a}{2} = \frac{P}{2S_cb}.$$

$$\frac{a}{2} = \frac{1200}{2 \times 300 \times 3} = \frac{2}{3} \text{ in.}$$

PROBLEMS

1. A cantilever beam of constant strength has a length of 6 ft., a constant width of 4 in., and a variable depth. It carries a uniformly distributed load of 800 lb. per linear ft. Compute the depth at the middle and at the fixed end if allowable $S = 1500$ lb./in.^{2}. *Ans.* 6.57 in.; 13.14 in.

2. A cantilever beam of constant strength has a length of 3 ft., a constant depth of 2 in., and a variable width. It carries a uniformly distributed load of 1800 lb. per linear ft. Compute the width at 1 ft., 2 ft., and 3 ft. from the free end if allowable $S = 18{,}000$ lb./in.^{2}.

Ans. $b_1 = 0.9$ in.; $b_2 = 3.6$ in.; $b_3 = 8.1$ in.

3. A cantilever beam of constant strength is 4 ft. long, 3 in. wide, and carries a load of 1600 lb. at the free end. Allowable $S_c = 450$ lb./in.^{2}; $S_s = 120$ lb./in.^{2}; $S = 1200$ lb./in.^{2}. Compute the depth at the free end and at each 1-ft. point along the beam. Compute also the extra length required.

Ans. $d_o = d_1 = 6.67$ in.; $d_2 = 8$ in.; $d_3 = 9.8$ in.; $d_4 = 11.31$ in.; $a/2 = 0.5926$ in.

4. A cast-iron cantilever beam of constant strength is to be 3 ft. long and is to carry a concentrated load of 2000 lb. at the free end and a uniformly distributed load of 600 lb. per linear ft. The section is to be rectangular, with the depth twice the width. Allowable $S = 3000$ lb./in.^{2}; $S_c = 12{,}000$ lb./in.^{2}; $S_s = 2500$ lb./in.^{2}. Compute the depth at the free end and at each 1-ft. point along the beam. Compute also the extra length required.

Ans. $d_o = 1.55$ in.; $d_1 = 4.8$ in.; $d_2 = 6.29$ in.; $d_3 = 7.47$ in.; $a/2 = 0.1075$ in.

5. A steel gear shift lever for an automobile is to be designed as a beam of constant strength 24 in. long, with a load of 40 lb. normal to the lever at the free end. Using allowable $S = 20{,}000$ lb./in.^{2}, compute the diameter at the fixed end, at the middle, and at a point 1 in. from the free end.

Ans. 0.788 in.; 0.625 in.; 0.273 in.

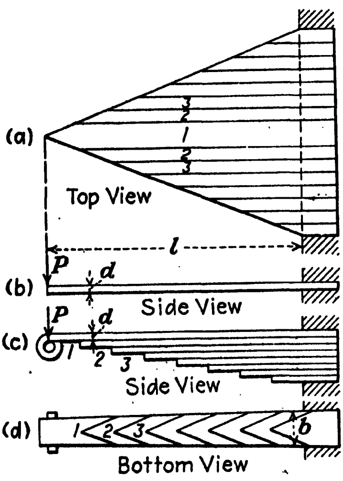


FIG. 144.

89. Leaf Springs.—The ordinary leaf spring is a modification of the beam of constant strength and constant depth. Figure 144(*a*) shows the top view and Fig. 144(*b*) the side view of a beam of this kind. If, now, the beam is sheared along the lines indicated in Fig. 144(*a*) and pieces 2, 2 are placed below piece 1, pieces 3, 3 below pieces 2, 2, and so on, the beam will take the form indicated in Figs. 144(*c*) and (*d*). The ends of the individual leaves are usually modified by being made wider and thinner in order to make a neater appearance. The beam which is composed of the separate leaves one below the other is just as strong as the beam with the leaves beside each other, as in Fig. 144(*a*). In the former case the load is transmitted from one leaf to the next in bearing. In the latter case the load is transmitted from one strip to the other in shear. The free end of the spring is modified for the attachment of the load. Springs of this type are often curved initially and straighten under load. The semi-elliptical spring may be considered as two cantilevers and the elliptical spring as four cantilevers.

If d is the thickness of the leaves, b the width of each leaf, and n the number of leaves, the equivalent width at the fixed end is $bn = B$, and the constant stress S at the outer fibers of each leaf is given by the equation,

$$S = \frac{6Pl}{bnd^2}.$$

PROBLEMS

1. A leaf spring 25 in. long is composed of seven leaves, each 2 in. wide and 0.2 in. thick. Compute the unit stress caused by a load of 600 lb. at the free end. *Ans.* 160,700 lb./in.^{2}.

2. A leaf spring 30 in. long is composed of twelve leaves, each 3 in. wide and 0.3 in. thick. If the allowable stress is 120,000 lb./in.^{2}, what is the allowable load? *Ans.* 2160 lb.

3. A leaf spring 20 in. long is to carry a load of 1000 lb. at the end with an allowable unit stress of 150,000 lb./in.^{2}. If the leaves are each 2.5 in. wide and 0.25 in. thick, how many leaves will be required? *Ans.* 6.

4. A spring 30 in. long is to be composed of ten leaves, each 3/8 in. thick. If the allowable stress is 125,000 lb./in.^{2}, what must be the width of the leaves if the spring is to carry a load of 4000 lb.? *Ans.* 4.096 in.

5. A spring 18 in. long is to be composed of 5 leaves, each 2 in. wide, and the allowable stress is 140,000 lb./in.^{2}. If the full length leaf is to be 1.5 times as thick as the other four, what is the required thickness for a load of 1000 lb.? *Ans.* $d_1 = 0.42$ in.; $d = 0.28$ in.

90. Deflection of Leaf Springs.—Since the leaf spring is equivalent to a beam of constant strength and constant depth, the differential equation from which the value of the deflection may be obtained becomes

$$M = EI\frac{d^2y}{dx^2} = \frac{SI}{c}.$$

$$E\frac{d^2y}{dx^2} = \frac{S}{c}.$$

Both S and c are constants, and c may be replaced by $d/2$. Let the origin of coordinates be taken at point O, Fig. 145.

$$E\frac{d^2y}{dx^2} = \frac{2S}{d}.$$

By integration,

$$E\frac{dy}{dx} = \frac{2S}{d}x + C_1.$$

$$\frac{dy}{dx} = 0 \text{ where } x = l, \text{ so } C_1 = -\frac{2S}{d}l.$$

$$E\frac{dy}{dx} = \frac{2S}{d}x - \frac{2S}{d}l.$$

$$Ey = \frac{S}{d}x^2 - \frac{2Sl}{d}x + C_2.$$

$$y = 0 \text{ where } x = l, \text{ so } C_2 = \frac{Sl^2}{d}.$$

$$Ey = \frac{S}{d}x^2 - \frac{2Slx}{d} + \frac{Sl^2}{d}.$$

The maximum deflection is at the free end where $x = 0$.

$$\text{Max. } y = \frac{Sl^2}{Ed}.$$

In terms of the load P and the moment of inertia I_m at the fixed end, since $M_m = -Pl$,

$$\text{Max. } y = -\frac{Pl^3}{2EI_m}.$$

This deflection is 50 per cent greater than the deflection of a cantilever beam with the same cross section throughout as that of the beam of constant strength at the fixed end, and with the same load P at the free end. The material it contains is only one-half as much.

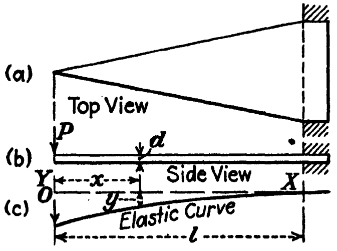


FIG. 145.

PROBLEMS

1. A leaf spring 25 in. long, composed of eight leaves, each 3 in. wide and 0.4 in. thick, carries a load of 2400 lb. Using $E = 30{,}000{,}000$ lb./in.^{2}, compute the deflection and the maximum unit stress.

Ans. 4.88 in.; 93,750 lb./in.^{2}.

2. A leaf spring 20 in. long composed of 12 leaves each 4 in. wide and ⅜ in. thick is designed so that it can have a deflection of 4 in. Compute the maximum load and the maximum stress, using E as 30,000,000 lb./in.^{2}.

Ans. 6330 lb.; 112,500 lb./in.^{2}.

3. A leaf spring 30 in. long is composed of 5 leaves, each 2 in. wide and ¼ in. thick. Using E as 30,000,000 lb./in.^{2}, compute the load which will deflect it 1 in. Compute the total load it will carry if the allowable unit stress is 120,000 lb./in.^{2}. *Ans.* 28.94 lb.; 417 lb.

91. Simple Beam of Constant Strength.—If a simple beam carries a concentrated load, each part of the beam from the load to either one of the supports is under the same condition of loading as a cantilever beam with a load at the end. In Fig. 146(a), the condition of support and loading of length a is the same as if the beam were fixed at the load P and loaded with an upward load of Pb/l at the end. The method of solution is the same as was used in Art. 88.

If a simple beam carries a uniformly distributed load, as shown in Fig. 146(b), the moment at any section at a distance of x from the left end is

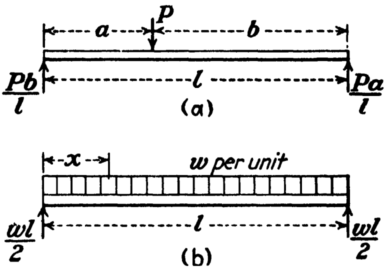


FIG. 146.

$$M = \frac{wlx}{2} - \frac{wx^2}{2}.$$

This expression for the moment must be used in the equation $M/S = I/c$ in order to determine the variable value of I/c necessary for a beam of constant strength. As in the case of a cantilever beam with a concentrated load, provision must be made for the bearing and shear at the ends.

EXAMPLE

A rectangular cast steel beam 6 ft. long, supported at the ends, is to carry a total uniformly distributed load of 7200 lb. If designed as a beam of constant strength with the depth twice the width, compute the depth at 1, 2, and 3 ft. from the end. Compute the depth just inside the bearing as determined by shear. Compute the total length of bearing at each end. Allowable S = 16,000 lb./in.^{2}; S_s = 8000 lb./in.^{2}; S_c = 20,000 lb./in.^{2}.

Solution.—Reaction R = 3600 lb. At 1 ft. from the end, $M_1 = 3600 \times 12 - 1200 \times 6 = 36{,}000$ in.-lb. Since $b = d/2$, $I/c = d^3/12$.

$$\frac{36{,}000}{16{,}000} = \frac{d_1^3}{12}.$$
$$d_1^3 = 27.$$
$$d_1 = 3 \text{ in.}$$

$$M_2 = 3600 \times 24 - 2400 \times 12 = 57{,}600.$$

$$\frac{57{,}600}{16{,}000} = \frac{d_2^3}{12}.$$

$$d_2^3 = 43.2.$$

$$d_2 = 3.51 \text{ in.}$$

$$M_3 = 3600 \times 36 - 3600 \times 18 = 64{,}800.$$

$$\frac{64{,}800}{16{,}000} = \frac{d_3^3}{12}.$$

$$d_3^3 = 48.6.$$

$$d_3 = 3.65 \text{ in.}$$

$$1.5 \times \frac{3600}{0.5d_o^2} = 8000.$$

$$d_o^2 = 1.35.$$

$$d_o = 1.16 \text{ in.}$$

$$b_o = 0.58 \text{ in.}$$

$$a \times 0.58 \times 20{,}000 = 3600.$$

$$a = 0.31 \text{ in.}$$

PROBLEMS

1. With the same allowable stresses as those given in the example above, design a cast steel beam of constant strength 6 ft. long, supported at the ends, to carry a total load of 40,000 lb. uniformly distributed, if the depth is three times the width.

Ans. $d_1 = 6.08$ in.; $d_2 = 7.11$ in.; $d_3 = 7.40$ in.; $d_o = 3.354$ in.; $a = 0.895$ in.

2. A square cast-iron beam of constant strength 4 ft. long, supported at the ends, is to carry a total uniformly distributed load of 24,000 lb. Using allowable $S = 3000$ lb./in.^{2}, $S_s = 3000$ lb./in.^{2}, and $S_c = 16{,}000$ lb./in.^{2}, compute the depth at the ends, at 1 ft. from the ends, and at the middle. Compute the length of bearing a.

Ans. $d_o = 2.45$ in.; $d_1 = 6$ in.; $d_2 = 6.6$ in.; $a = 0.306$ in.

92. Plate Girders and Box Girders.—Plate girders and box girders as usually constructed are approximately beams of constant strength. The riveted plate girder consists of a web plate a, Fig. 147(a), four flange angles b, and two flange plates c, all of which extend the full length of the girder. The welded plate girder consists of a web plate and two flange plates, fillet-welded, as shown in Fig. 147(b). Near the ends, where the moment is small, the section modulus is larger than needed. At some certain distance from each end, the section modulus is just large enough, while between these points it is not large enough. The added strength necessary is obtained by riveting or welding on an extra pair of plates extending between these points. With these added plates, the section modulus may be large enough for the load. If it is not, however, another computation is made as before, continuing until the section modulus is large enough for the load at the middle. For accurate computation, the areas of the rivet holes in the flanges in riveted girders should be deducted in making computation of the section modulus. If the rivets in the compression flange completely filled the holes, the areas of rivet holes in that flange would not need to be deducted, but it is common practice to deduct the areas of rivet holes in both flanges. If only the areas in the tension flange are deducted, the centroid of the area of the cross section is no longer at the center of the web, and must therefore be located. The section modulus for the tension flange is the one which governs.

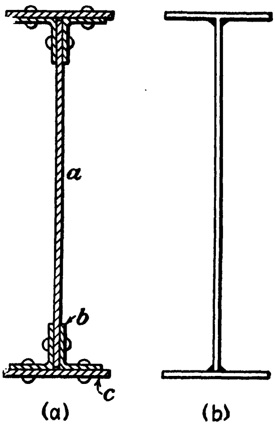


FIG. 147.

Box girders are usually composed of channels and plates or I-beams and plates, either riveted or welded together. The material in box girders is not used so efficiently as is that in plate girders, but extra width for the support of the load is more easily obtained. As with plate girders, the areas of the rivet holes in the flanges of riveted box girders should be deducted in the computation of the section modulus.

EXAMPLE

A plate girder 80 ft. long, supported at the ends, is composed of a web plate 48 by ½ in., four flange angles each 6 by 6 by ½ in. spaced 48½ in. back to back, and two flange plates each 16 by ½ in. extending the full length. It is to support a uniformly distributed load of 3600 lb. per linear ft. with an allowable stress of 18,000 lb./in.^{2}. Compute the minimum lengths of the additional flange plates if they are also 16 by ½ in. Rivets are ¾ in. in diameter in $^{13}\!/_{16}$-in. punched holes. There are two rows of rivets joining the flange plates and the flange angles, and one row of rivets joining the flange angles and the web plate, with the center of the row 3½ in. from the backs of the angles.

Solution.—The total load is 288,000 lb., so each reaction is 144,000 lb. The load w per linear inch is 300 lb. The moment is

$$M = 144{,}000x - 150x^2.$$

The moment of inertia of the solid section as computed from values given in the table of properties of sections is $I = 26{,}007.3$ in.^{4}. For punched holes for ¾-in. rivets, the width of area to be deducted is ⅞ in. For four rivet holes in the flanges,

$$I = 4 \times \frac{0.875}{12} + 4 \times 0.875 \times 24.25^2 = 2058.5 \text{ in.}^4.$$

For two rivet holes in the web,

$$I = 2 \times 1.5 \times \frac{0.875^3}{12} + 2 \times 1.3125 \times 20.75^2 = 1130.5 \text{ in.}^4.$$

The net moment of inertia is

$$I = 26{,}007.3 - 2058.5 - 1130.5 = 22{,}818.3 \text{ in.}^4.$$

The section modulus $I/c = 22{,}818.3/24.75 = 922.0$ in.^{3}.

$$\begin{aligned} 144{,}000x - 150x^2 &= 18{,}000 \times 922.0. \\ x &= 134 \text{ in. or } 826 \text{ in.} \end{aligned}$$

These two values of x are the distances from the left end at which the section modulus of the girder is just sufficient to carry the load. Between these two points the section modulus is not large enough and the section must be increased by adding flange plates at the top and bottom. The minimum length of the first pair of plates is therefore $l_1 = 826 - 134 = 692$ in., or 57 ft. 8 in.

With the added pair of plates, the moment of inertia and the section modulus are both increased.

$$\begin{aligned} I_2 &= 22{,}818.3 + 8906.6 = 31{,}724.9 \text{ in.}^4. \\ \frac{I_2}{c_2} &= \frac{31{,}724.9}{25.25} = 1256.4 \text{ in.}^3. \\ 144{,}000x - 150x^2 &= 18{,}000 \times 1256.4. \\ x &= 198 \text{ in. or } 762 \text{ in.} \\ l_2 &= 564 \text{ in., or } 47 \text{ ft. } 0 \text{ in.} \\ I_3 &= 31{,}724.9 + 9266.4 = 40{,}991.3 \text{ in.}^4. \\ \frac{I_3}{c_3} &= \frac{40{,}991.3}{25.75} = 1591.9 \text{ in.}^3. \\ 144{,}000x - 150x^2 &= 18{,}000 \times 1591.9. \\ x &= 282 \text{ in. or } 678 \text{ in.} \\ l_3 &= 396 \text{ in., or } 33 \text{ ft. } 0 \text{ in.} \\ I_4 &= 40{,}991.3 + 9633.3 = 50{,}624.6 \text{ in.}^4. \\ \frac{I_4}{c_4} &= \frac{50{,}624.6}{26.25} = 1928.6 \text{ in.}^3. \\ 144{,}000x - 150x^2 &= 18{,}000 \times 1928.6. \end{aligned}$$

The solution of this equation gives imaginary values for x, so no more plates are needed. This could have been determined also by solving for the bending stress at the middle with the three added pairs of plates.

$$M = \frac{288,000}{8} \times 960 = 34,560,000.$$

$$S = \frac{34,560,000}{1928.6} = 17,920 \text{ lb./in.}^2.$$

PROBLEMS

1. A box girder 40 ft. long, supported at the ends, is composed of two 15-in. 50-lb. channels spaced 10 in. back to back, and two 18- by ½-in. plates welded to the channels and extending the full length. The girder is to carry a uniformly distributed load of 3000 lb. per linear ft. and a concentrated load of 40,000 lb. at the middle with an allowable unit stress of 18,000 lb./in.^{2}. Compute the minimum lengths of the additional flange plates if they are also 18 by ½ in.

Ans. 30 ft. 4 in.; 23 ft. 10 in.; 15 ft. 10 in.; 4 ft. 2 in.

2. A plate girder 60 ft. long, supported at the ends, is composed of a web plate 48 by ⅜ in., four flange angles each 6 by 6 by ¾ in. spaced 48½ in. back to back, and two flange plates each 16 by ¾ in. Rivets are ⅞ in. in diameter in 15⁄16-in. drilled holes. There are two rows of rivets connecting the flange plates and the flange angles, and one row of rivets connecting the flange angles and the web plate, with the center of the row 3½ in. from the backs of the angles. The girder is to support a uniformly distributed load of 6000 lb. per linear ft. with an allowable stress of 18,000 lb./in.^{2}. Compute the minimum lengths of the additional flange plates if they are also to be 16 by ¾ in. The areas of the rivet holes are to be deducted in both tension and compression flanges. *Ans.* 34 ft. 2 in.; 13 ft. 6 in.

CHAPTER XI

RESILIENCE

93. Work and Resilience in Bars.—During the application of a load to an elastic body, motion ensues in the direction of the force, and therefore work is done upon the body by the force. Since in elastic deformations the distance varies directly with the force, the work done is the product of the distance and the average force. The average force is one-half of the sum of the initial and the final values of the force. If the initial value of the force is zero, the average force is one-half of the final value.

Since an elastic body recovers its original size and shape when its deforming load is removed, it has the capacity of doing work. That is, when a portion of the load is removed, the body does mechanical work by moving back the remaining part of the load. When stored in the elastic body, the work done upon it is in the form of elastic potential energy, and is called *resilience.* The resilience of a unit volume of any material when stressed to its elastic limit is called the *modulus of resilience* of the material. The modulus of resilience is the *measure* of the resilience of the material, and is the maximum amount of work which may be stored up in a unit volume of the material and then recovered without loss.

If the unit of volume is the cubic inch and the stress is measured in pounds per square inch, resilience is given in units of inch-pounds per cubic inch. As the stress increases from zero to S, the deformation increases from zero to δ. The average force is $S/2$, so the work done is $S/2 \times \delta$. Since $\delta = S/E$, the work done per cubic inch is $S/2 \times S/E = S^2/2E$. Since the work done upon the body is equal to the elastic resilience stored in the body,

$$\text{Unit elastic resilience} = \frac{S^2}{2E}.$$

When the value of S becomes the elastic limit of the material, this expression gives the modulus of resilience, as stated above.

If the value of S exceeds the elastic limit, some of the work is used up in overcoming the frictional resistance of the molecules of the material as they slide over each other to take a permanent deformation, and hence cannot be recovered in the form of mechanical work.

If the stress in a body is uniform throughout the body, the total elastic energy is given by the product of the unit resilience and the volume of the body.

EXAMPLE 1

Compute the total amount of work stored in a wrought-iron bar 1 in. in diameter and 8 in. long, while the total load is changed from 10,000 to 25,000 lb. Use $E = 27{,}000{,}000$ lb./in.^{2}.

Solution:

$$S_1 = \frac{10{,}000}{0.7854} = 12{,}730 \text{ lb./in.}^2.$$

$$S_2 = \frac{25{,}000}{0.7854} = 31{,}830 \text{ lb./in.}^2.$$

The initial unit resilience $= 12{,}730^2/2 \times 27{,}000{,}000 = 3.00$ in.-lb./cu. in.
The final unit resilience $= 31{,}830^2/2 \times 27{,}000{,}000 = 18.78$ in.-lb./cu. in.
The increase in unit resilience $= 18.78 - 3.00 = 15.78$ in.-lb./cu. in.
The total increase in resilience $= 15.78 \times 8 \times 0.7854 = 99.15$ in.-lb.

EXAMPLE 2

A steel cable with a cross-section area of 0.4 sq. in. and a length of 1000 ft. supports a mine cage weighing 4800 lb. Using $E = 24{,}000{,}000$ lb./in.^{2}, compute the total resilience in the cable due to its own weight and the weight of the mine cage.

Solution.—At the lower end, $S = 12{,}000$ lb./in.^{2}. At a distance of x in. above the lower end, $S = 12{,}000 + 0.284x$. The resilience per cu. in. $= 1/2E(12{,}000 + 0.284x)^2$. Since the area is 0.4 sq. in., the resilience per inch of length $= 1/5E(12{,}000 + 0.284x)^2$. The resilience in dx length $= 1/5E(12{,}000 + 0.284x)^2dx$. The integral of this expression between the limits of 0 and 12,000 gives the total resilience.

$$U = \frac{1}{5E}\left(12{,}000^2\int_0^{12{,}000} dx + 6816\int_0^{12{,}000} x dx + 0.08065\int_0^{12{,}000} x^2 dx\right).$$

$$U = 18{,}877 \text{ in.-lb.}$$

PROBLEMS

1. A steel bar 0.5 in. in diameter and 8 in. long is loaded with a total load of 7000 lb. in tension. $E = 29{,}000{,}000$ lb./in.^{2}. Compute the unit resilience, the total resilience, and the amount of elongation.

Ans. 21.9 in.-lb./cu. in.; 34.4 in.-lb.; 0.00984 in.

2. A block of southern yellow pine 2 in. square and 10 in. long placed on end and loaded with a compressive load of 28,000 lb., shortened 0.026 in. in a gage length of 8 in. Compute E and the unit resilience.

Ans. 2,154,000 lb./in.^{2}; 11.38 in.-lb./cu. in.

3. Steel railroad rails 39 ft. long weighing 120 lb./yd. are laid in a track with spaces of 0.12 in. between ends when the temperature is 45° F. $E = 29{,}000{,}000$ lb./in.^{2}. Compute the total amount of resilience in each rail when the temperature is 120° F. *Ans.* 4260 in.-lb.

4. A steel cable with a cross-section area of 0.6 sq. in. and a length of 750 ft. supports an elevator weighing 3600 lb. Using $E = 18{,}000{,}000$ lb./in.^{2}, compute the total resilience in the cable due to the elevator and its own weight. Compute also the unit resilience at the upper end.

Ans. 8027 in.-lb.; 2.03 in.-lb/cu. in.

94. Impact Stresses in Bars.—In all of the discussion of bars in tension and short blocks in compression, it was assumed that the loads were applied slowly and uniformly. This is practically the case if the load is increased by small increments from zero to its maximum value P. The average force is $P/2$ and the work done in deforming the bar an amount e, if the elastic limit is not exceeded, is

$$U = \frac{Pe}{2}.$$

If now a weight W is held in contact with a bar and then is suddenly released to cause tension or compression in the bar, the entire amount of the weight W acts through the distance e through which the bar is deformed, and the work done by gravity on the weight is

$$U = We.$$

As the load comes on the bar, the pressure on the bar increases from zero at the instant the load is released up to a maximum value P as the load comes to rest at the bottom of its travel. The work done against the variable resistance is $Pe/2$ as before if the elastic limit of the material is not exceeded. Some of the work done by gravity on the weight is lost in air resistance, some is used in deforming the weight itself and some is used in deforming the supports of the bar, but ordinarily the greater part of it is used against the resistance of the bar. Assuming that all of the work done on the weight by gravity is stored in the bar in elastic resilience,

$$\frac{Pe}{2} = We.$$
$$P = 2W.$$

The maximum pressure of a suddenly applied load approaches a value twice as great as the static weight.

After coming to rest at the bottom of its travel, since the upward pressure on the weight is twice the downward force, the weight vibrates back up to its original position (approximately). If there were no losses, the vibration would continue indefinitely, but on account of the losses, the weight soon comes to rest in its static position with a deflection which is one-half as much as the maximum deflection.

If now the weight W is dropped from a height h before it comes in contact with the bar and then produces a tensile or compressive deformation of e, the total work done by gravity is $W(h + e)$. If the losses mentioned above are neglected and if the elastic limit is not exceeded,

$$W(h + e) = \frac{Pe}{2}.$$

As before, P is the maximum pressure of the weight on the bar. In terms of the maximum stress S, assuming the value of E to be the same for impact as for static loads,[1] this becomes

$$W(h + e) = \frac{S^2}{2E}Al.$$
$$S^2 = \frac{2EW}{Al}(h + e).$$

Since W/A is the static unit stress S_0 and since $e = Sl/E$,

$$S^2 - 2S_0S - \frac{2ES_0h}{l} = 0.$$
$$S = S_0\left(1 + \sqrt{1 + \frac{2hE}{S_0l}}\right).$$

The value of e is small compared with h, and if e is neglected, it will be noticed that the unit stress caused by an impact load varies inversely as the square root of the length of the bar. If the length of the bar is doubled, the impact stress is approximately 0.707 times as much.

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[1] There is some evidence that the value of E for impact loads is higher than for static loads, but no authoritative experiments to determine accurate values have as yet been made.

If a weight W moving with a velocity v is stopped by a bar in tension or compression, the work done is equal to $Wv^2/2g$. If v is in units of feet per second and g, the acceleration of gravity, in feet per second per second, the work is in units of foot-pounds. Neglecting the losses at the point of impact, the work done by the moving body as it is brought to rest is equal to the resilience in the bar.

PROBLEMS

1. A weight of 50 lb. drops a distance of 8 in. and strikes axially on a cylinder of cast iron 6 in. in diameter and 10 in. high. Using the value of E as 15,000,000 lb./in.^{2}, compute the unit stress caused.

Ans. 6520 lb./in.^{2}.

2. If the value of E is 18,000,000 lb./in.^{2}, what unit stress is caused in a steel cable 1200 ft. long and 0.8 sq. in. in cross section by the fall of a weight of 2000 lb. through a distance of 15 ft.? What will be the elongation caused?

Ans. 36,130 lb./in.^{2}; 28.91 in.

3. If the value of E is 14,000,000 lb./in.^{2}, how long a bar 1 in. in diameter will be necessary to take the impact of a 30-lb. weight striking with a velocity of 40 ft. per sec. without exceeding a stress of 35,000 lb./in.^{2}?

Ans. 21.75 ft.

95. Helical Springs.—A helical spring is formed by winding a wire or rod, usually circular in cross section, around a cylinder in one layer, the axis of the rod taking the form of a helix. The rod is annealed to eliminate initial stresses, then slipped off the cylinder and tempered.

A spring which is to be used in tension only is usually wound with the coils touching each other. A spring which is to be used in compression must be wound with spaces between the coils. Such a spring can, of course, be used in either tension or compression.

The end coil of a spring which is to be used in tension is bent across the end of the coil and formed into a loop or hook to which to attach the load. The end coil of a spring which is to be used in compression is tapered, the tapered part being in contact with the next coil and the whole end formed so that the plate through which the load is applied is in contact around the complete circumference.

Figure 148 represents a helical spring with the coils close enough together so that the axis of the coil is very nearly in the plane of any cross section of the wire or rod of which the coil is made, as at A. An axial load P upon the spring in either tension or compression causes a torque in the rod which is approximately equal to PR, R being the radius of the coil, the distance from the axis of the coil to the axis of the rod. In addition to this torsional shearing stress there is a direct shearing stress at every section due to the load P, but this direct stress is so small compared to the torsional stress that it is usually neglected.

All of the principles of torsion as derived in Arts. 32 and 33 apply to the helical spring.

$$S_s = \frac{Tr}{J} = \frac{PRr}{J}.$$

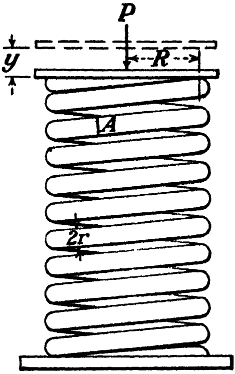


FIG. 148.

The lengthening or shortening y of the coil is equal to the radius of the coil multiplied by the angle of twist of the rod.

$$y = R\theta.$$

If n is the number of free turns of the rod, $l = 2\pi Rn$.

If the coil spring represented in Fig. 148 is loaded in compression until the coils are in contact, the maximum stress in torsion is reached. Any further increase in the load merely compresses the wires of the coil laterally. The end attachments for coil springs used in tension are usually so designed that the elongation of the spring cannot exceed the amount which will produce the allowable stress in the spring. Since with the usual construction there is no danger of exceeding the allowable unit stress for which a coil spring is designed, even though the load for which it was designed is exceeded, the allowable unit stress may closely approach the elastic limit.

The scale of a spring is the load necessary to cause unit deflection in the spring. A "600-pound" spring is a spring which will be deflected 1 inch by a load of 600 pounds.

EXAMPLE

The spring shown in Fig. 148 has 8.5 free turns of steel rod 1.2 in. in diameter wound on a 4-in. cylinder with spaces of 0.9 in. between coils. Compute the load P which will compress the coils completely together. Compute the unit shearing stress S_s. $E_s = 12{,}000{,}000$ lb./in.^{2}.

Solution.—$R = 2.6$ in. With 8.5 free turns, there are 8.5 spaces, so the deflection when the coils touch is $y = 8.5 \times 0.9 = 7.65$ in.

$$y = R\theta = \frac{RTl}{E_sJ}.$$

$$7.65 = \frac{2.6 \times P \times 2.6 \times 2 \times \pi \times 2.6 \times 8.5 \times 2}{12{,}000{,}000 \times \pi \times 0.6^4}.$$

$$P = 19{,}910 \text{ lb.}$$

$$S_s = \frac{Tr}{J} = \frac{19{,}910 \times 2.6 \times 0.6 \times 2}{\pi \times 0.6^4} = 152{,}500 \text{ lb./in.}^2$$

PROBLEMS

1. A helical spring is made of 24 turns of steel wire 0.224 in. in diameter wound upon a 1-in. cylinder with the coils in contact. Compute the lengthening of the spring and the maximum shearing stress caused by a load of 250 lb. $E_s = 12{,}000{,}000$ lb./in.^{2}. *Ans.* 2.91 in.; 69,300 lb./in.^{2}.

2. A helical spring is composed of five free turns of steel wire 1 in. in diameter, with 5.4 in. outside diameter of coil. Compute the scale of the spring. If the spring is to be used in compression, compute the spacing necessary between the coils so that the maximum shearing stress will not exceed 125,000 lb./in.^{2} when the coils are compressed together.
Ans. 3520 lb./in.; 0.634 in.

3. If $E_s = 6{,}000{,}000$ lb./in.^{2} for brass wire, compute the outside diameter of coil necessary so that 30 free turns of wire $\frac{1}{8}$ in. in diameter will have a scale of 10 lb./in. Through what length may this coil be stretched safely if S_s is not to exceed 50,000 lb./in.^{2}? *Ans.* 0.973 in.; 4.5 in.

4. A helical spring has six free turns of steel rod 1.4 in. in diameter, with a radius of coil of 3.2 in. and spaces of 0.8 in. between coils. Compute the load and the unit stress when the coils are compressed completely together if $E_s = 12{,}000{,}000$ lb./in.^{2}. *Ans.* 17,590 lb.; 104,400 lb./in.^{2}.

96. Resilience in Torsion.—If a torque which is uniformly increased from zero to T is applied to a shaft to produce an angular twist of θ radians, the work of the external couple is given by the product of the average torque and the angle of twist.

$$U = \frac{T\theta}{2} = \frac{T^2l}{2E_sJ}.$$

This external work is equal to the internal resilience, and in terms of the maximum unit shearing stress S_s is

$$U = \frac{S_s^2 Jl}{2E_s r^2}.$$

In the solid circular shaft or bar,

$$U = \frac{S_s^2}{4E_s}\pi r^2 l.$$

If the bar is formed into a helical spring, the resilience of the spring in terms of the external work is given by the expression,

$$U = \frac{P}{2}y,$$

in which P is the maximum load on the spring and y is the deflection.

Since $\pi r^2 l$ is the volume of the spring, the quantity $S_s^2/4E_s$ is the average unit resilience in the spring, and is one-half as much as the unit resilience in a bar subjected to a unit shearing stress of S_s throughout. On account of the fact that the value of the shearing modulus of elasticity E_s is much smaller than that of the tensile and compressive modulus of elasticity E, the average elastic unit resilience in a coil spring is much greater than that in a leaf spring made from material of the same grade.

PROBLEMS

1. Compute the maximum unit resilience and the total resilience in a shaft 4 in. in diameter and 20 ft. long which is transmitting 800 hp. at 450 r.p.m. $E_s = 12{,}000{,}000$ lb./in.^{2}.
Ans. 3.31 in.-lb./cu. in.; 4995 in.-lb.

2. Compute the total resilience in a shaft, 2 in. outside diameter, 1.6 in. inside diameter, and 8 ft. long which is transmitting 180 hp. at 1600 r.p.m. $E_s = 12{,}000{,}000$ lb./in.^{2}. *Ans.* 216 in.-lb.

3. Compute the total resilience and the average unit resilience in the spring described in the example, Art. 95.
Ans. 76,200 in.-lb.; 485 in.-lb./cu. in.

4. A helical spring is to take the shock of a body weighing 800 lb. falling a total distance of 8 in. without exceeding a stress of 120,000 lb./in.^{2}. What is the weight of the lightest steel spring that can be used? *Ans.* 6.06 lb.

97. Resilience in Beams.—A beam which has loads applied to it suffers elastic deformation and contains elastic energy or resilience. The amount of this resilience may be obtained either in terms of the work of the external load or in terms of the internal resisting stress.

If a concentrated load P is applied to a beam gradually to cause a deflection of y_1 at the point of application, the average force is $P/2$ and the work done is equal to the product of the average force and the total distance y_1 through which the load has moved. The resilience stored in the beam is equal in amount to the external work, and if it is represented by U,

$$U = \frac{P}{2}y_1.$$

Since the direction of application of the load and the deflection caused by it are always in the same direction, the work is always positive and the sign of the deflection need not be considered.

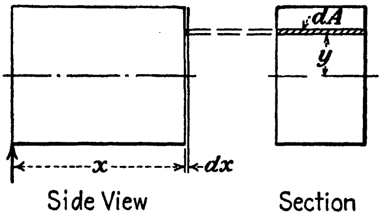


FIG. 149.

If a uniformly distributed load of w per unit length is applied to a beam gradually, each differential part of the weight, wdx, moves a different distance y. The work done by each differential weight is $wydx/2$, and the total work done by the load is the sum of these differential quantities.

$$U = \frac{w}{2}\int ydx.$$

The value of the variable deflection y is obtained from the equation of the elastic curve, after which the integration may be performed.

In terms of the variable internal resisting stress S, the unit elastic resilience at any point in a beam is $S^2/2E$. The stress S varies both with the distance from the neutral axis and with the distance x along the beam. If y represents the distance of any point in the cross section from the neutral axis, as in Art. 45, the variable stress $S = My/I$, and the unit resilience $= M^2y^2/2EI^2$.

In the differential volume $dAdx$, shown in Fig. 149, the resilience is

$$dU = \frac{M^2y^2}{2EI^2}dAdx.$$

The total resilience in the beam is

$$U = \frac{1}{2E}\int\int\frac{M^2}{I^2}y^2dAdx.$$

Since

$$\int y^2dA = I,$$

$$U = \frac{1}{2E}\int\frac{M^2}{I}dx.$$

EXAMPLE 1

By both methods, derive the expression for the total resilience in a cantilever beam of uniform cross section with a concentrated load P at the end, shown in Fig. 150.

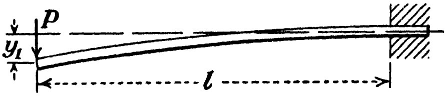


FIG. 150.

Solution.—By the method of external work: By Art. 59, the deflection at the end is

$$y_1 = \frac{Pl^3}{3EI}.$$

$$U = \frac{Py_1}{2}.$$

$$U = \frac{P^2l^3}{6EI}.$$

By the method of internal work: Since the moment of inertia I is constant and the variable moment $M = Px$,

$$U = \frac{P^2}{2EI}\int_0^l x^2dx.$$

$$U = \frac{P^2l^3}{6EI}.$$

In terms of the maximum stress S, since $M = Pl = SI/c$,

$$U = \frac{S^2Il}{6Ec^2}.$$

If the section is rectangular, the average unit resilience is $S^2/18E$.

EXAMPLE 2

By both methods, derive the expression for the total resilience in a fixed beam of uniform cross section, with a uniformly distributed load

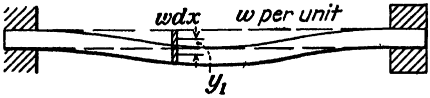


FIG. 151.

Solution.—By the method of external work: The beam is shown in Fig. 151. By Art. 73, the variable deflection is

$$y = \frac{w}{24EI}(x^4 - 2lx^3 + l^2x^2).$$

$$U = \frac{w}{2}\int y\,dx = \frac{w^2}{48EI}\int_0^l (x^4 - 2lx^3 + l^2x^2)dx.$$

$$U = \frac{w^2l^5}{1440EI} = \frac{W^2l^3}{1440EI}.$$

By the method of internal work: Since the moment of inertia I is constant, and the moment

$$M = \frac{w}{12}(6lx - l^2 - 6x^2),$$

$$U = \frac{w^2}{288EI}\int_0^l (l^4dx - 12l^3xdx + 48l^2x^2dx - 72lx^3dx + 36x^4dx).$$

$$U = \frac{w^2l^5}{1440EI} = \frac{W^2l^3}{1440EI}.$$

In terms of the maximum stress S, since the maximum moment

$$M = \frac{Wl}{12} = \frac{SI}{c},$$

$$U = \frac{S^2Il}{10Ec^2}.$$

If the section is rectangular, the average unit resilience is $S^2/30E$.

PROBLEMS

1. Derive the expressions for the total resilience and the average unit resilience in a cantilever beam of rectangular cross section with a uniformly distributed load. *Ans.* $\frac{W^2l^3}{40EI}; \frac{S^2}{30E}$.

2. Derive the expression for the total resilience in a simple beam with a uniformly distributed load. *Ans.* $\frac{W^2l^3}{240EI}$.

3. Derive the expression for the total resilience in a simple beam with a load P at a distance a from one end and b from the other. Use the method of external work. *Ans.* $\frac{P^2a^2b^2}{6EIl}$.

4. Derive the expression for the average unit resilience in a beam with a circular cross section, fixed at both ends, with a load P at the middle.

Ans. $\frac{S^2}{24E}$.

98. Resilience in a Leaf Spring.

98. Resilience in a Leaf Spring.—The ordinary leaf spring is equivalent to a beam of constant strength and constant depth, as shown in Fig. 152. As was shown in Art. 90, the deflection at the end due to the load P is $y_1 = Pl^3/2EI_m$. In this expression, I_m is the moment of inertia of the cross section at the fixed end.

The resilience in the beam is

$$U = \frac{Py_1}{2} = \frac{P^2l^3}{4EI_m}.$$

In terms of the maximum stress S, since $M = Pl = SI_m/c$,

$$U = \frac{S^2I_ml}{4Ec^2}.$$

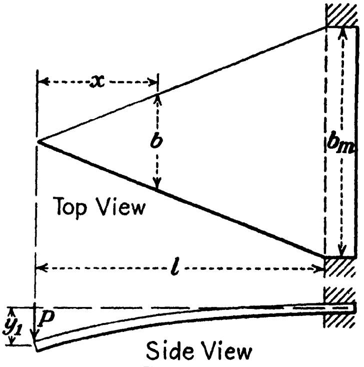


FIG. 152.

In terms of the dimensions of the beam, this becomes

$$U = \frac{S^2}{6E}\frac{b_mld}{2}.$$

Since $b_mld/2$ is the volume of the beam, the average unit resilience is $S^2/6E$. This is three times as much as the average unit resilience in a rectangular beam with a constant cross section loaded similarly, but only one-third as much as that in a bar subjected to a direct stress of S throughout.

EXAMPLE

Compute the size of leaves of a spring 24 in. long to carry a load of 1800 lb. with a deflection of 6 in. and a unit stress of 125,000 lb./in.^{2} in bending. Use $E = 30,000,000$ lb./in.^{2}.

Solution.—The external work $Py_1/2$ is equal to the internal resilience, $S^2/6E \times$ volume.

$$1800 \times 3 = \frac{125,000^2}{6 \times 30,000,000} \times 12b_md.$$

From $M = SI/c$,

$$1800 \times 24 = \frac{125{,}000 b_m d^2}{6}.$$
$$d = 0.4 \text{ in.}$$
$$b_m = 12.96 \text{ in.}$$

With six leaves, each leaf would be 2.16 in. wide. With five leaves, each leaf would be 2.592 in. wide.

PROBLEMS

1. A leaf spring 20 in. long is composed of 10 leaves, each 4 in. wide and 0.35 in. thick. Using E as 30,000,000 lb./in.^{2}, compute the resilience in the spring due to a load of 3000 lb. *Ans.* 4198 in.-lb.

2. Assuming eight leaves, compute the size of leaves of a spring 30 in. long to carry a load of 5000 lb. with a deflection of 10 in. and a unit stress of 150,000 lb./in.^{2} in bending. Use $E = 30{,}000{,}000$ lb./in.^{2}. *Ans.* 3.7 by 0.45 in.

3. A leaf spring is to take the shock of a body weighing 800 lb. falling a total distance of 8 in. without exceeding a stress of 150,000 lb./in.^{2}. What is the weight of the lightest steel spring that can be used? Compare with the result of Problem 4, Art. 96. *Ans.* 14.55 lb.

99. Impact Stresses in Beams.—If a load on a beam is suddenly applied and all of the work done by gravity upon the load is assumed to be stored in the beam in elastic energy, the deformation and the stress will be twice as great as for the same static load.

If a load W drops from a height h upon a beam and causes a deflection of y, the total work done by gravity upon the load is $W(h + y)$. If P is the maximum pressure of the load on the beam, the work done against this resistance as it increases from zero to P is $Py/2$. If the losses due to air resistance, deformation of the load W itself, and deformation of the supports of the beam are neglected,

$$W(h + y) = \frac{Py}{2}.$$

If P_1 is the load to cause a deflection of unity, the pressure $P = P_1 y$.

$$W(h + y) = \frac{P_1 y^2}{2}.$$

In many cases the beams which form parts of machines are subjected to blows which cause impact stresses. If W is the weight of a body and v is the velocity with which it strikes a blow on a beam, the energy which it delivers to the beam is $Wv^2/2g$, g being the acceleration of gravity. If W is in pounds, v in feet per second, and g in feet per second per second, the energy will be given in foot-pounds. This must be reduced to inch-pounds when used with the expressions for deflection.

EXAMPLE 1

A block weighing 0.6 lb. falls 5 in. and strikes the middle of a wooden beam 1.2 in. wide, 0.25 in. deep, and 30 in. long, supported at the ends. Using the value of E as 2,100,000 lb./in.^{2}, compute the deflection and the stress.

Solution.—The load which will deflect the beam unit distance is

$$P_1 = \frac{48EI}{l^3}.$$
$$P_1 = 5.83 \text{ lb.}$$
$$0.6(5 + y) = \frac{5.83y^2}{2}.$$
$$y = 1.122 \text{ in.}$$
$$P = P_1y = 6.55 \text{ lb.}$$
$$S = \frac{Mc}{I} = 3930 \text{ lb./in.}^2.$$

EXAMPLE 2

A steel rod 1 in. in diameter projects 8 in. from a solid support. With what velocity must a 5-lb. sledge strike it transversely at the end to cause a stress of 40,000 lb./in.^{2} if $E = 28{,}000{,}000$ lb./in.^{2}?

Solution.—If v is the velocity of the sledge in feet per second, the energy of the sledge in inch-pounds is

$$\frac{5 \times 12 \times v^2}{2 \times 32.2} = 0.9315v^2.$$
$$y = \frac{Sl^2}{3Ec} = \frac{40{,}000 \times 64}{3 \times 28{,}000{,}000 \times 0.5} = 0.061 \text{ in.}$$
$$P_1 = \frac{3EI}{l^3} = 8053 \text{ lb.}$$

The energy of the beam is

$$\frac{P_1y^2}{2} = 14.96 \text{ in.-lb.}$$
$$0.9315v^2 = 14.96.$$
$$v^2 = 16.$$
$$v = 4 \text{ ft./sec.}$$

PROBLEMS

1. A body weighing 300 lb. drops 10 ft. and strikes the middle of a 20-in. 100-lb. I-beam, 20 ft. long, fixed at both ends. Neglecting the losses of energy, compute the maximum stress and the maximum deflection. Use $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 39,900 lb./in.^{2}; 0.33 in.

2. The fiber stress at elastic limit of a hickory beam tested in impact bending at the Forest Products Laboratory, Madison, Wisconsin, was 22,900 lb./in.^{2}, and its modulus of elasticity was 2,350,000 lb./in.^{2}. If a spring board of this material is 16 in. wide, 1.8 in. deep, and 7 ft. long, from what height can a body weighing 175 lb. drop upon the free end of the board without exceeding the elastic limit? *Ans.* 12 ft. 2 in.

CHAPTER XII

COMBINED STRESSES

100. Combined Flexural and Direct Stresses.—If in addition to its transverse loading a beam is subjected to direct axial loading, the resultant stress at any point in a cross section is equal to the algebraic sum of the stresses produced by the two loadings independently. If P is the direct axial load and A is the area of the cross section, the direct unit stress at any point in the cross section is

$$S_1 = \frac{P}{A}.$$

If M is the flexural moment at any section caused by the transverse loads upon the beam, y the distance of any point from the neutral axis of the cross section, and I the moment of inertia of the cross section, the flexural unit stress at that point is

$$S_2 = \frac{My}{I}.$$

The total stress S at any point is the algebraic sum of the two.

$$S = \frac{P}{A} \pm \frac{My}{I}.$$

The maximum and minimum stresses are the ones usually desired, and these are given by the expression,

$$S = \frac{P}{A} \pm \frac{Mc}{I}.$$

EXAMPLE 1

A steel eyebar used as the bottom chord of a bridge is 2 in. wide, 12 in. deep, and 30 ft. long. If it carries a tensile load of 750,000 lb., compute the maximum and minimum unit stresses due to the direct load and its own weight.

Solution:

$$S_1 = \frac{P}{A} = \frac{750{,}000}{24} = 31{,}250 \text{ lb./in.}^2 \text{ T.}$$

Neglecting the heads, $W = \frac{1}{6} \times 1 \times 30 \times 490 = 2450$ lb.

$$S_2 = \frac{2450 \times 360 \times 6}{8 \times 2 \times 144} = 2300 \text{ lb./in.}^2.$$

At the bottom of the cross section at the middle between supports,

$$\text{Max. } S = 31{,}250 + 2300 = 33{,}550 \text{ lb./in.}^2 \text{ T.}$$

At the top of the cross section at the middle between supports,

$$\text{Min. } S = 31{,}250 - 2300 = 28{,}950 \text{ lb./in.}^2 \text{ T.}$$

EXAMPLE 2

A concrete wall 8 ft. high and 3 ft. thick, monolithic with a concrete base at the bottom, has water 6 ft. high on one side and none on the other. Using the weight of concrete as 150 lb./cu. ft. and that of water as 62.5 lb./cu. ft., compute the stresses at the bottom of the wall, both inside and outside, in pounds per square foot.

Solution.—$S_1 = 8 \times 150 = 1200$ lb./ft.^{2}. This is the vertical compressive unit stress of the base on the wall at all points. With a 1-ft. length of the wall as the unit, the horizontal pressure of the water on the wall at the bottom is $6 \times 62.5 = 375$ lb./ft.^{2}. The horizontal pressure varies uniformly from this value at the bottom to zero at a point 6 ft. up from the bottom. The average horizontal unit pressure on the wall is 187.5 lb./ft.^{2}, and the total horizontal pressure on the unit length of wall is $6 \times 187.5 = 1125$ lb. The center of pressure of this horizontal force is at a point 2 ft. from the bottom, and the moment of the horizontal pressure about the neutral axis of the cross section of the wall at the bottom is

$$M = 1125 \times 2 = 2250 \text{ ft.-lb.}$$

$$S_2 = \frac{2250 \times 6}{1 \times 9} = 1500 \text{ lb./ft.}^2.$$

At the inner face, the stress S_2 is tension, so

$$S = 1500 - 1200 = 300 \text{ lb./ft.}^2 \text{ T.}$$

At the outer face, the stress S_2 is compression, so

$$S = 1500 + 1200 = 2700 \text{ lb./ft.}^2 \text{ C.}$$

PROBLEMS

1. A wooden cantilever beam, 3 in. wide, 12 in. deep, and 6 ft. long carries a load of 250 lb. per linear foot including its own weight, and an axial compressive load of 4000 lb. Compute the maximum and minimum stresses at the fixed end. *Ans.* 861 lb./in.^{2} C.; 639 lb./in.^{2} T.

2. A vertical wooden post 8 in. square and 6 ft. high fixed at the lower end, carries a vertical load of 6000 lb. on top and a side thrust of 1200 lb. parallel to one side of the post at a point 2 ft. from the upper end. Compute the maximum and minimum stresses. *Ans.* 769 lb./in.^{2} C.; 581 lb./in.^{2} T.

3. A steel shaft 3 in. in diameter and 2 ft. long fixed at one end, carries an axial compressive load of 10,000 lb. Compute the side thrust at the free end which will reduce the stress to zero at one side. *Ans.* 156 lb.

4. Solve Example 2 above if the water is the full height of the wall on one side and 2 ft. high on the other.

Ans. 2300 lb./ft.² T.; 4700 lb./ft.² C.

101. Eccentric Loading.—If a block is subjected to a force which is not applied along its axis, the single force produces the two effects of direct stress and flexural stress. In Fig. 153, P represents the resultant of the compressive forces applied at the top of the block $KLMN$, at a distance e from the axis of the block. Since a force may be replaced by a force through any given point and a couple, the force P may be replaced by an axial force at C and a couple whose moment is Pe. The unit stress at N is

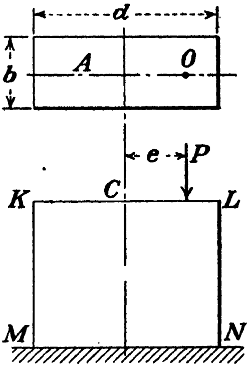


FIG. 153.

$$S = \frac{P}{A} + \frac{Pec}{I},$$

and that at M is

$$S = \frac{P}{A} - \frac{Pec}{I}.$$

If the stress at M is zero,

$$\frac{P}{A} = \frac{Pec}{I}.$$

For the rectangular section represented,

$$\frac{P}{bd} = \frac{6Pe}{bd^2}.$$

$$e = \frac{d}{6}.$$

For an eccentricity larger than $d/6$ in either direction parallel to d, or larger than $b/6$ in either direction parallel to b, the stress at any point on the opposite side is reversed in sign. Since brick or stone masonry laid in lime mortar has practically no tensile strength, the resultant load on a wall or rectangular pier which is made of such masonry should not fall outside of the middle third.

If the resultant of the applied load is not acting in the plane of either principal axis, it must remain within the rhombus $ABCD$, Fig. 154, if there is to be no reversal of stress at the opposite corner. For if the resultant is acting at any point outside of the rhombus, as at O, it can be resolved into two components, each of which is on one of the principal axes. If one of these components is acting at B, the edge of the middle third on the longer axis, the other component will be acting at point E, which is outside of the middle third on the shorter axis, and will therefore cause a reversal of stress at the opposite corner.

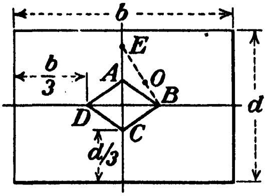


FIG. 154.

The expression $S = P/A \pm Mc/I$ applies also to the ordinary U clamp and to the hook if the base of the U and the shank of the hook are straight. If they are curved, as is usually the case, the expression applies only approximately.

PROBLEMS

1. Derive the expression for the maximum eccentricity e without reversal of stress for a circular cross section. *Ans.* $e = \frac{r}{4}$.

2. If k is the radius of gyration of a cross section of any shape, derive the expression for the maximum eccentricity e without reversal of stress, in terms of the radius of gyration. *Ans.* $e = \frac{k^2}{c}$.

3. Compute the maximum eccentricity e along the diagonal of a square cross section without reversal of stress at the opposite corner.

Ans. $e = \frac{b}{6} \times 0.707$.

4. A concrete foundation wall 30 in. thick has a load of 24,000 lb. per linear ft. of wall applied along a line 12 in. from the front face of the wall. Compute the unit stresses at the front and at the rear faces due to the load.

Ans. 15,360 lb./ft.^{2} C.; 3840 lb./ft.^{2} C.

5. A hollow concrete pier is 3 ft. square on the outside and 2 ft. square on the inside. If the resultant load is 50,000 lb. applied at a point 8 in. from the center on a line through the center parallel to one of the sides, what are the maximum and minimum stresses due to the load? Compute the eccentricity without reversal of stress. Compute the eccentricity along a diagonal without reversal of stress.

Ans. 19,230 lb./ft.^{2} C.; 770 lb./ft.^{2} C.; 8.67 in.; 6.13 in.

6. The shank of a hook is 3 in. in diameter and the line of action of the load is 4 in. from the center of the shank. Compute the maximum and minimum stresses in the shank due to a load of 10,000 lb.

Ans. 16,500 lb./in.^{2} T.; 13,670 lb./in.^{2} C.

7. The shank of a hook is triangular, with a base of 2 in. and an altitude of 3 in. The base of the triangle is on the inside of the shank, and the line of action of the load is 2 in. from the face of the shank. If the safe stress in tension is 18,000 lb./in.^{2}, and that in compression is 24,000 lb./in.^{2}, what is the safe load that the hook can carry? *Ans.* 6545 lb.

8. A load of 1000 lb. is supported on a bracket with a T-section as shown in Fig. 155. Compute the stresses at M and N due to the load.

Ans. 1950 lb./in.^{2} T.; 4890 lb./in.^{2} C.

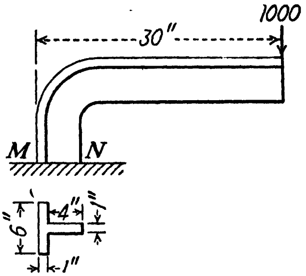
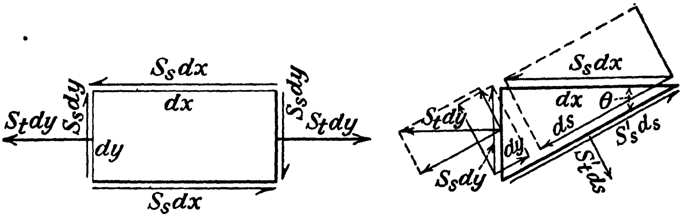


FIG. 155.

102. Resultant of Direct and Shearing Stresses.

—Figure 156(a) represents, as a free body, a parallelepiped of length dx, height dy, and width unity, upon which is exerted a tensile stress of S_t per unit area in the X direction and a direct shearing stress of S_s per unit area in the Y direction. The direct vertical shearing stress induces a horizontal shearing stress of equal unit intensity in the X direction, as was shown in Art. 16.

Let this parallelepiped be divided by a diagonal plane, and let the differential prism above and to the left of this diagonal plane be taken as the free body, as shown in Fig. 156(b). S_s' is the unit shearing stress on the diagonal plane ds, and S_t' is the unit tensile stress on the diagonal plane ds.



(a) (b)

FIG. 156.

The forces on the end and top surfaces are resolved into their components parallel to ds and perpendicular to ds. Since the forces acting upon the prism are in equilibrium, the summation of forces parallel to the plane ds gives,

$$S_s'ds = S_t dy \cos\theta - S_s dy \sin\theta + S_s dx \cos\theta.$$

If this equation is divided by ds, and if dy/ds is replaced by $\sin\theta$ and dx/ds by $\cos\theta$, it becomes,

$$S_s' = S_t \sin\theta\cos\theta - S_s \sin^2\theta + S_s \cos^2\theta.$$

In terms of the angle 2θ,

$$S_s' = \frac{S_t}{2}\sin 2\theta + S_s \cos 2\theta.$$

The summation of forces normal to the plane ds gives,

$$\begin{aligned} S_t'ds &= S_t dy \sin\theta + S_s dy \cos\theta + S_s dx \sin\theta. \\ S_t' &= S_t \sin^2\theta + 2S_s \sin\theta\cos\theta. \\ S_t' &= \frac{S_t}{2}(1 - \cos 2\theta) + S_s \sin 2\theta. \end{aligned}$$

If all of the forces on the parallelepiped in Fig. 156 are reversed, the same expressions are obtained, except that S_t' becomes S_c' and the shearing stress S_s' is reversed in direction. If the shearing forces alone are reversed, the angle θ is below the X axis.

PROBLEMS

1. A block has a direct tensile stress of 1600 lb./in.^{2} and a shearing stress of 600 lb./in.^{2} at right angles to the direction of the tensile stress. Compute the unit shearing stress S_s' on a plane at an angle of 15° with the direction of the direct tensile stress, and the unit tensile stress S_t' normal to the same plane. *Ans.* 920 lb./in.^{2}; 407 lb./in.^{2}.

2. Solve Problem 1 for an angle of 45°. *Ans.* 800 lb./in.^{2}; 1400 lb./in.^{2}.

103. Maximum Resultant Shearing Stress.—If the first derivative of S_s' with respect to θ in the expression,

$$S_s' = \frac{S_t}{2}\sin 2\theta + S_s \cos 2\theta,$$

is equated to zero, the value of θ to make S_s' a maximum is obtained.

$$\frac{dS_s'}{d\theta} = S_t \cos 2\theta - 2S_s \sin 2\theta = 0.$$

$$\text{Tan } 2\theta = \frac{\left(\frac{S_t}{2}\right)}{S_s}.$$

Angles with positive values of the tangent are in the first or third quadrant.

Figure 157 shows a right triangle in the first quadrant, the legs of which represent $S_t/2$ and S_s. The angle opposite $S_t/2$ is then 2θ and the hypotenuse is $\sqrt{(S_t/2)^2 + S_s^2}$. If the values of sin 2θ and cos 2θ from this triangle are substituted in the general equation for S_s', it becomes,

$$\text{Max. } S_s' = \pm \sqrt{(S_t/2)^2 + S_s^2}.$$

There are two values of $2\theta_s$, 180° apart, for which $\tan 2\theta = (S_t/2)/S_s$, hence there are two values of θ_s, 90° apart. The planes along which this maximum value of S_s' occurs are the planes OA and OB, Fig. 157.

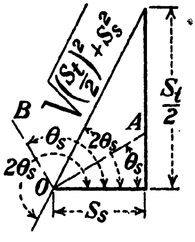


FIG. 157.

EXAMPLE

Compute the direction and magnitude of the maximum resultant shearing stress in the block described in Problem 1, Art. 102.

Solution:

$$\frac{S_t}{2} = 800.$$

$$S_s = 600.$$

$$\tan 2\theta_s = \frac{800}{600} = 1.33.$$

$$2\theta_s = 53° 08' \text{ or } 233° 08'.$$

$$\theta_s = 26° 34' \text{ or } 116° 34'.$$

$$\text{Max. } S_s' = \sqrt{800^2 + 600^2} = 1000 \text{ lb./in.}^2$$

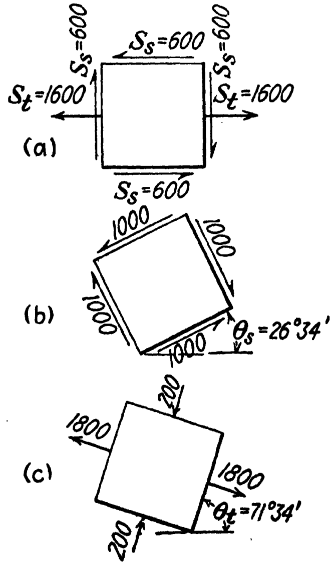


FIG. 158.

Figure 158(*a*) represents a block with unit dimensions upon which are acting the direct tensile stress, the direct vertical shearing stress, and the induced horizontal shearing stress. Figure 158(*b*) shows the direction and magnitude of the maximum resultant shearing stresses.

PROBLEMS

1. Compute the direction and magnitude of the maximum resultant shearing stress in a block due to a direct horizontal tensile stress of 18,000 lb./in.^{2} and a vertical shearing stress of 3000 lb./in.^{2}.

Ans. 35° 47'; 9487 lb./in.^{2}.

2. Solve Problem 1 if the shearing stress is only 300 lb./in.².
Ans. 44° 03′; 9005 lb./in.².

104. Maximum Resultant Normal Stress.

—If the first derivative of S_t' with respect to θ in the expression,

$$S_t' = \frac{S_t}{2}(1 - \cos 2\theta) + S_s \sin 2\theta,$$

is equated to zero, the value of θ to make S_t' a maximum is obtained.

$$\frac{dS_t'}{d\theta} = S_t \sin 2\theta + 2S_s \cos 2\theta.$$

$$\operatorname{Tan} 2\theta = -\frac{S_s}{\left(\frac{S_t}{2}\right)}.$$

Angles which have negative values of the tangent are in the second or fourth quadrant.

Figure 159 shows a right triangle in the second quadrant, the legs of which represent S_s and $-S_t/2$. The hypotenuse of the triangle is $\sqrt{(S_t/2)^2 + S_s^2}$. If the values of sin 2θ and cos 2θ from this triangle are substituted in the general equation for S_t', it becomes,

$$\text{Max. } S_t' = \frac{S_t}{2} + \sqrt{\left(\frac{S_t}{2}\right)^2 + S_s^2}.$$

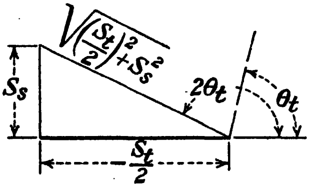


FIG. 159.

If the right triangle in the fourth quadrant is used, S_s is negative and $S_t/2$ is positive. With the values of sin 2θ and cos 2θ from this triangle substituted in the equation for S_t', it becomes,

$$\text{Min. } S_t' = \frac{S_t}{2} - \sqrt{\left(\frac{S_t}{2}\right)^2 + S_s^2}.$$

Since $\sqrt{(S_t/2)^2 + S_s^2}$ is the expression for the maximum value of the resultant shearing stress S_s', the two expressions for S_t' may be written together.

$$\text{Maximum or minimum } S_t' = S_t/2 \pm \text{max. } S_s'.$$

Since the maximum value of S_s' is always greater than $S_t/2$, the minimum value of S_t' is always negative in sign, which signifies compression.

A comparison of Figs. 157 and 159 shows that the angle $2\theta_t$ is 90° larger than $2\theta_s$, so θ_t is 45° larger than θ_s.

EXAMPLE

Compute the direction and magnitude of the maximum and minimum resultant tensile stresses in the block described in Problem 1, Art. 102.

Solution:

$$\frac{S_t}{2} = 800.$$

$$S_s = 600.$$

$$\tan 2\theta_t = -\frac{600}{800} = -0.75.$$

$$2\theta_t = 143^\circ\,08' \text{ or } 323^\circ\,08'.$$

$$\theta_t = 71^\circ\,34' \text{ or } 161^\circ\,34'.$$

$$\text{Max. } S_t' = 800 + 1000 = 1800 \text{ lb./in.}^2.$$

$$\text{Min. } S_t' = 800 - 1000 = -200 \text{ lb./in.}^2.$$

The negative sign indicates that the stress is compression. Figure 158(*c*) shows the direction and magnitude of the maximum and minimum resultant tensile stresses.

The value of θ_t could have been obtained from the result of the example of Art. 103.

$$\theta_t = \theta_s + 45^\circ.$$

PROBLEMS

1. Compute the direction and magnitude of the maximum and minimum resultant tensile stresses in the block described in Problem 1, Art. 103.

Ans. 80° 47′ and 170° 47′; 18,487 lb./in.^{2} T.; 487 lb./in.^{2} C.

2. A block has a direct horizontal compressive stress of 20,000 lb./in.^{2} and a vertical shearing stress of 4000 lb./in.^{2}. Compute the direction and magnitude of the maximum and minimum resultant compressive stresses.

Ans. 79° 06′; 169° 06′; 20,770 lb./in.^{2} C.; 770 lb./in.^{2} T.

105. Combined Flexural and Shearing Stresses in Beams.—If a comparatively short, deep beam is loaded heavily enough to develop the allowable flexural stresses in tension and compression, very large shearing stresses are developed.

The largest tensile or compressive stress occurs at the outer fibers of the beam where the shearing stress is zero, while the largest shearing stress occurs at the neutral surface where the tension and compression are zero.

It is easily possible that even though the direct flexural and shearing stresses are below the allowable values, the maximum resultant shearing stress at some point may exceed the allowable stress. In the case of timber beams, however, the direction of this maximum resultant shearing stress is partly across the grain and so may still be allowable.

It is very seldom that the maximum resultant tensile or compressive stress will exceed the value of the direct flexural stress in the outer fibers. Computation of the maximum resultant stresses at two or three points will usually be sufficient to show how they vary across the beam.

PROBLEM

1. Using the safe bending stress for southern yellow pine as 1600 lb./in.^{2} and the safe shearing stress as 110 lb./in.^{2} as given in Table V, compute the safe load at the middle of a simple beam 8 in. wide, 24 in. deep, and 5 ft. long. Compute the direction and magnitude of the maximum resultant shearing and tensile stresses at points 2 in., 6 in., and 10 in. from the bottom, in a section just at the left of the load.

Ans. $P = 28{,}160$ lb. At 2 in., $\theta_s = 40° 50'$, $\theta_t = 85° 50'$, $S_s' = 231$ lb./in.^{2}, $S_t' = 460$ lb./in.^{2}. At 6 in., $\theta_s = 29° 30'$, $\theta_t = 74° 30'$, $S_s' = 160$ lb./in.^{2}, $S_t' = 298$ lb./in.^{2}. At 10 in., $\theta_s = 11° 35'$, $\theta_t = 56° 35'$, $S_s' =$ 116 lb./in.^{2}, $S_t' = 162$ lb./in.^{2}.

106. Combined Flexural and Shearing Stresses in Shafts.—In Chap. IV it was assumed that the shafts were subjected to torque alone. In practice, however, shafts are usually subjected to bending stresses due to their own weight, to the weight of pulleys, to the tension in belts, and sometimes to centrifugal forces.

There are, therefore, three different direct stresses in a shaft which transmits power: (1) shearing stresses due to torsion; (2) tensile and compressive stresses due to bending; and (3) shearing stresses due to bending. Except in extremely short shafts, heavily loaded flexurally, the latter shearing stress need not be considered, since it is zero where the flexural tensile and compressive stresses are greatest.

The torsional shearing stress reaches the same maximum value at all of the outer fibers, so the maximum resultant shearing and tensile stresses will occur at the outer fibers on the tensile side of the shaft at the danger sections. On the opposite or compressive side of the shaft, the maximum resultant shearing and compressive stresses will occur. As the shaft rotates, these points of maximum stress rotate also and thus produce regular reversals of stress. Material subjected to reversals of stress has an endurance limit somewhat lower than the elastic limit as determined by ordinary static tests.

In terms of the direct torque T and the direct bending moment M, the equivalent torque T' and the equivalent bending moment M' are given by the following equations:

$$T' = \sqrt{M^2 + T^2}.$$
$$M' = \tfrac{1}{2}(M + T').$$

PROBLEMS

1. A shaft 2 in. in diameter is subjected to a torque of 600 ft.-lb. and a bending moment of 500 ft.-lb. Compute the direct stresses and the maximum resultant stresses.

Ans. $S_s = 4580$ lb./in.^{2}; $S_t = 7640$ lb./in.^{2}; $S_s' = 5960$ lb./in.^{2}; $S_t' = 9780$ lb./in.^{2}.

2. A steel shaft 3 in. in diameter extends 30 in. beyond its end support and on the free end carries a pulley 3 ft. in diameter weighing 100 lb. On one side of the pulley the tension in the belt is 600 lb. vertically downward. On the other side of the pulley the tension is 250 lb., also vertically downward. Compute the maximum resultant shearing and tensile stresses.

Ans. $S_s' = 5680$ lb./in.^{2}; $S_t' = 11{,}230$ lb./in.^{2}.

3. A steel shaft 3 in. in diameter and 25 ft. long is supported at the ends. At a point 2 ft. from the left end is a pulley 2 ft. in diameter weighing 80 lb., upon which are downward vertical belt pulls of 600 and 150 lb. At a point 1 ft. from the right end is a pulley 3 ft. in diameter weighing 150 lb. upon which are upward vertical belt pulls of 420 and 120 lb., the torque being opposite to that at the left end. Compute the maximum resultant shearing and tensile stresses. *Ans.* $S_s' = 6080$ lb./in.^{2}; $S_t' = 12{,}075$ lb./in.^{2}.

CHAPTER XIII

EULER'S COLUMN FORMULA

107. Classes of Columns.—Members of structures and machines which are subjected to axial compressive loads are called posts, struts, or columns if their length is several times as great as their least lateral dimension. If l is the length of a column and k is the least radius of gyration of its cross section, the ratio l/k is called the *slenderness ratio* of the column. The slenderness ratio of columns is the basis for comparison of columns of different lengths and of different sizes and shapes of cross section.

Compression members for which the value of l/k is less than 30 are not usually called columns, but for the purpose of classification may be called *short* columns. Instead of using a column formula for short columns, it is customary to specify a reduced value of the allowable unit stress.

Compression members for which the value of l/k is between 30 and 160 are called *medium length* columns. Practically all columns used in such structures as bridges and buildings fall in this class.

Compression members for which the value of l/k is greater than 160 are called *long* columns. Some secondary members, such as wind and sway bracing in bridges, and some members used in machines fall in this class.

PROBLEMS

1. Compute the length of a bar 2 in. in diameter for which $l/k = 30$; for which $l/k = 160$. *Ans.* 15 in.; 80 in.

2. Compute the maximum value of l/k for a column 3 by 8 in. in cross section and 16 ft. long. *Ans.* 222.

3. A steel pipe is 1 in. in diameter outside and $\frac{3}{4}$ in. in diameter inside. For what length of pipe will the slenderness ratio equal 160? *Ans.* 50 in.

4. Between what lengths are 5-in. 10-lb. I-beams classed as medium length columns? *Ans.* 19.5 in. to 8 ft. 8 in.

108. Euler's Formula for Long Columns.—Figure 160 shows a column which is subjected to an axial load P causing it to buckle laterally. At cross section B, the stress on the concave side of the column is given by the same expression as was derived in Art. 101 for eccentric loads, $S = P/A + Mc/I$. For long columns, the term Mc/I becomes very large compared with the term P/A. If the term P/A is neglected, the value of the buckling load P may be obtained in terms of the modulus of elasticity E and the dimensions of the column.

In order that the same notation may be used as was used for beams, the X axis is taken in the direction of the length of the column, positive upward, and the Y axis is taken horizontal at the lower end of the column, positive to the left. As in the case of the beam which is convex upward, the moment in the column shown is negative and the deflection y is positive. If the column were buckled to the right, the moment would be positive and the deflection negative. In either case, the moment and the deflection are opposite in sign, or $M = -Py$.

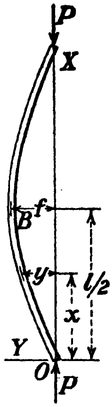


FIG. 160.

The moment equation in terms of the second derivative of y with respect to x becomes,

$$EI\frac{d^2y}{dx^2} = -Py.$$

If this equation is multiplied by dy, it becomes,

$$EIdy\frac{d^2y}{dx^2} = -Pydy.$$

The expression $dy\frac{d^2y}{dx^2}$ may be written $\frac{dy}{dx}d\left(\frac{dy}{dx}\right)$.

$$EI\int\frac{dy}{dx}d\left(\frac{dy}{dx}\right) = -P\int ydy.$$

$$\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 = -\frac{P}{2}y^2 + C_1.$$

At the middle, $dy/dx = 0$ and $y = f$.

$$C_1 = \frac{P}{2}f^2.$$

$$EI\left(\frac{dy}{dx}\right)^2 = P(f^2 - y^2).$$

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\sqrt{f^2 - y^2}.$$

$$\int \frac{dy}{\sqrt{f^2 - y^2}} = \sqrt{\frac{P}{EI}} \int dx.$$

$$\sin^{-1} \frac{y}{f} = \sqrt{\frac{P}{EI}}\, x + C_2.$$

At the lower end, $y = 0$ and $x = 0$.

$$\sin^{-1} 0 = C_2.$$

$$C_2 = 0 \text{ or } n\pi,\ n \text{ being any integer.}$$

With the value of $C_2 = 0$,

$$\sin^{-1} \frac{y}{f} = \sqrt{\frac{P}{EI}}\, x.$$

At the upper end, $y = 0$ and $x = l$.

$$\sin^{-1} 0 = \sqrt{\frac{P}{EI}}\, l.$$

$$\sqrt{\frac{P}{EI}}\, l = 0 \text{ or } n\pi,\ n \text{ being any integer.}$$

If the value of zero is used, either $P = 0$ or $l = 0$.
If the value of $n = 1$ is used,

$$P = \frac{\pi^2 EI}{l^2}. \tag{1}$$

If the value of $n = 2$ is used,

$$P = \frac{4\pi^2 EI}{l^2}. \tag{2}$$

The curve into which the column bends is a sine curve. If $n = 1$, the column of length l bends into one sine curve as shown in Fig. 160 and the buckling load P is given by Equation (1) above. If $n = 2$, the column of length l bends into two sine curves, as shown in Fig. 161. This result is accomplished by means of the lateral supports on each side at the middle point D. In terms of the full length l the buckling load P is four times as great as it would be if the column were bent into one sine curve, as shown by Equation (2) above.

In terms of l_1, the length of the sine curve, the buckling load becomes,

$$P = \frac{\pi^2 EI}{l_1^2},$$

the same form as Equation (1).

If the moment of inertia I is replaced by its equivalent Ak^2, Equation (1) above reduces to,

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}.$$

It should be noted that the quantity P/A is not the maximum unit stress, but is the *average unit stress*, or what might be called the *unit load.*

It is found that a very slight increase in the load, above the value which causes a slight buckling, will increase the deflection greatly. The fiber stress increases with the deflection, and when it has passed the yield point of the material the column will fail with no further increase of the load. The buckling load, therefore, is taken as the ultimate load.

Since the fiber stress reached when failure begins is only the yield point of the material, the factor of safety permissible is much lower than that in the case of beams or of bars in tension. Instead of values of 4 or 5 for ordinary structural work, the values of 2, 2.5, or 3 may be used safely.

PROBLEMS

1. Compute the buckling load on a wooden column 2 by 6 in. in cross section and 10 ft. long, if $E = 1{,}200{,}000$ lb./in.^{2}. *Ans.* 3290 lb.

2. Compute the buckling load on a steel wire 0.109 in. in diameter and 22 in. long, if $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 4.1 lb.

3. A piece of steel pipe is 0.540 in. outside diameter, 0.364 in. inside diameter, and 8 ft. long. Compute the buckling load, if $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 103 lb.

109. End Conditions of Columns.—In the derivation of Euler's formula for long columns in Art. 108, it was assumed that the ends of the column were free to turn or rotate as the column deflected. Such columns may be called *round-end* columns, although there is some small resistance to the turning.

If a column is held rigidly at the ends so that it cannot turn, as shown in Fig. 162(a), it is called a *fixed-end* column. By symmetry, the tangent at the middle point C remains vertical.

From A to B the column is concave to the left, with the moment and the curvature decreasing from a maximum at A to zero at B. At B, the column changes curvature, being concave to the right from B to C, with the moment and the curvature increasing from zero at B to a maximum at C, the same in amount as that at A, but opposite in sign. Curve BC, if rotated 180° about point B, would be superimposed upon curve AB.

Since there is zero moment at B and at D, the portion BD of the column is under free bending the same as the column shown in Fig. 160. If l_1 is the length of the sine curve BD, the buckling load becomes

$$P = \frac{\pi^2 EI}{l_1^2}.$$

If l is the length of the entire column, $l_1 = l/2$, and

$$P = \frac{4\pi^2 EI}{l^2}.$$

It will be seen that this equation is the same as Equation (2), Art. 108, and applies to the case in which the column bends into two sine curves, as Figs. 161 and 162(a).

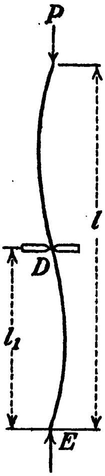
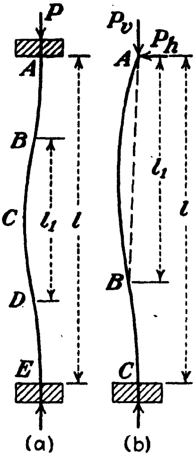


FIG. 161.



(a) (b)

FIG. 162.

Figure 162(b) represents another possible end condition, the lower end being fixed and the upper end free to turn, but held laterally directly above the lower end. Since the moment is zero at A and also at B, the point of counterflexure, the line of action of the resultant pressure at A must be in the direction AB. This requires that there must be a horizontal component P_h of the load at A.

The solution of the differential equation gives length $AB = 0.7l$ (approximately). If the length of the sine curve $l_1 = 0.7l$, Euler's formula for this case becomes,

$$P = \frac{2\pi^2 EI}{l^2} \text{ (approx.)}.$$

Ends perfectly free to turn and ends absolutely fixed are both ideal conditions which can only be approximated in actual columns. Even hemispherical, cylindrical, and knife-edge bearings used in laboratory tests offer some resistance to turning. The ordinary cylindrical pin used in connecting rods of engines and in columns of bridges gives still more resistance.

Absolutely fixed ends are also impossible, because all materials are elastic and yield to a certain extent as the load is applied.

Square-end columns are intermediate between those with ends free to turn and those with ends fixed. For large values of l/k, square-end columns approach round-end columns in behavior, while for low values of l/k they approach fixed-end columns in behavior.

PROBLEMS

1. Compute the buckling load on a wooden column 2 by 8 in. in cross section and 20 ft. long if it is supported laterally at the middle point and the value of E is 1,400,000 lb./in.^{2}. *Ans.* 5120 lb.

2. Compute the buckling load on a 5-in. 10-lb. I-beam 15 ft. long, if it is fixed at the lower end and round at the upper end. Use $E = 29{,}000{,}000$ lb./in.^{2}. *Ans.* 21,200 lb.

3. If a rectangular column of length l and cross-section dimensions b and d is supported laterally at the middle in the direction of dimension d, but is unsupported in the other direction, what must be the ratio of b to d so that it is equally strong in the two directions? *Ans.* $b = 2d$.

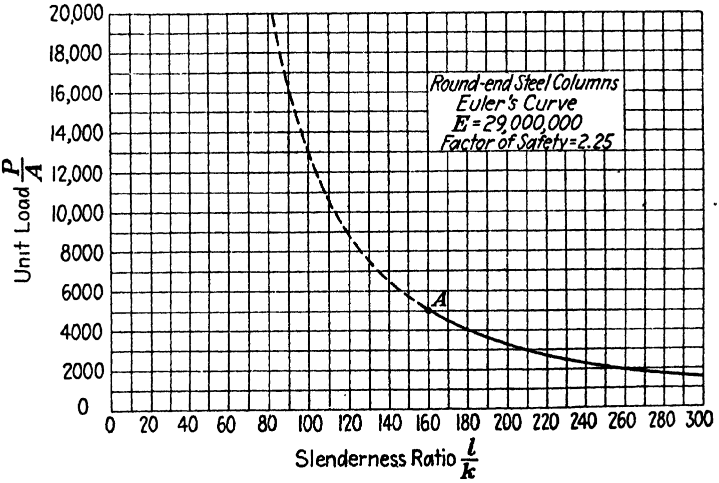


FIG. 163.

110. Limits for Use of Euler's Formula.—With a factor of safety of 2.25 and a value of E of 29,000,000 pounds per square inch, the safe unit load for steel columns with round ends is given by,

$$\frac{P}{A} = \frac{127{,}200{,}000}{\left(\frac{l}{k}\right)^2}.$$

Figure 163 is a diagram on which are plotted the corresponding values of the safe load P/A and the slenderness ratio l/k, for round-end steel columns. The curve is infinite in length in both directions. For large values of l/k, it is valid indefinitely. For small values of l/k it is not valid, since in its derivation only the effect of the flexural moment was considered. As the direct compressive component of the stress becomes an appreciable part of the total stress, the error in using Euler's formula also becomes appreciable, and the smaller the value of l/k the larger the error.

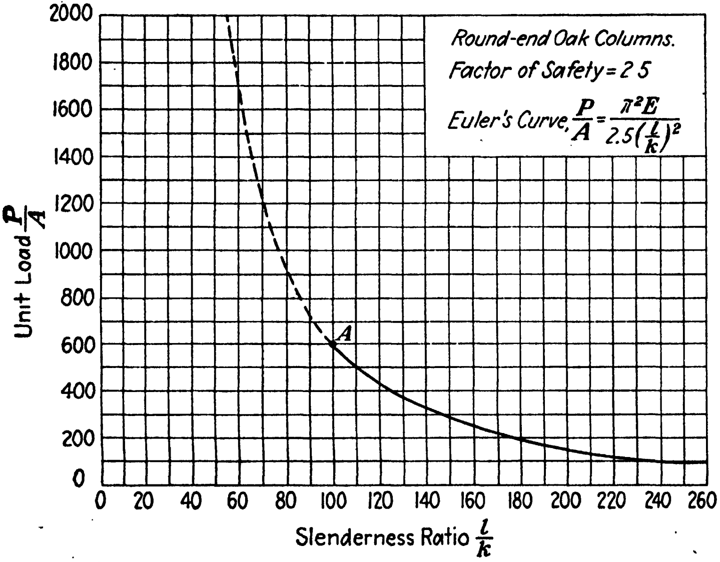


FIG. 164.

Euler's formula should not be used for columns with values of l/k less than about 160.

Figure 164 is a diagram on which are plotted the corresponding values of the safe load P/A and the slenderness ratio l/k for round-end oak columns with a factor of safety of 2.5 and a value of E of 1,500,000 lb./in.^{2}. This curve may be used for values of l/k as low as 100.

PROBLEMS

1. On an extension of the diagram of Fig. 163, plot Euler's curve for values of l/k up to 500.

2. Using the value of E as 2,000,000 lb./in.^{2}, and a factor of safety of 3, plot Euler's curve for wooden columns for values of l/k between 80 and 300.

111. Investigation of Columns by Euler's Formula.—Problems in the investigation of columns are of two kinds. In one, the dimensions and the loading of the column are given to compute the factor of safety. In the other, the dimensions of the column and the factor of safety are given to compute the safe load.

Computation of the value of l/k must always be made to determine if Euler's formula applies. As shown in Art. 110, if the value of l/k is less than 160 for steel columns, or less than 100 for wooden columns, Euler's formula should not be used. For columns of any other material, diagrams of P/A and l/k should be plotted to determine the lower limit of l/k for which such formula should be used. As will be explained in Chap. XV, this limiting value of l/k will be given by the point on the curve at which a straight line drawn through the safe value of P/A for short blocks will be tangent to the curve.

EXAMPLE

Compute the length of a 12-in. 31.8-lb. I-beam above which Euler's formula may be used.

Solution.—From the table of properties of I-beams, the minimum value of k is 1.01 in.

$$\frac{l}{k} = 160.$$

$$l = 160 \times 1.01 = 161.6 \text{ in.} = 13.5 \text{ ft.}$$

PROBLEMS

1. Compute the length of 2- by 4-in. oak columns above which Euler's formula may be used. *Ans.* 4.8 ft.

2. A loaded scaffolding weighing 2000 lb. is supported by four oak columns, each 2 by 4 in. by 10 ft. Assuming round-end conditions and using $E = 1{,}500{,}000$ lb./in.^{2}, compute the factor of safety. *Ans.* 5.48.

3. An 8-in. 18.4-lb. I-beam 20 ft. long, fixed at the lower end and round at the upper end, carries a load of 15,000 lb. $E = 29{,}000{,}000$ lb./in.^{2}. Compute the factor of safety. *Ans.* 2.52.

4. Compute the safe load on a wooden column 10 by 12 in. by 60 ft., assuming round-end conditions. Use $E = 1{,}200{,}000$ lb./in.^{2} and a factor of safety of 3. *Ans.* 7600 lb.

5. A horizontal steel connecting rod 7 ft. long has an elliptical cross section, with the vertical axis of the ellipse 2 in. and the horizontal axis 1.5 in. Using E as 30,000,000 lb./in.^{2}, and considering the rod as fixed-end type in the horizontal direction and round-end type in the vertical direction, compute the safe load in compression with a factor of safety of 2.5. *Ans.* 9890 lb.

6. Compute the safe load on a steel water pipe 1.05 in. in diameter outside, 0.824 in. in diameter inside, and 12 ft. long, when used as a column with fixed ends. Use a factor of safety of 2.25 and a value of $E = 29,000,000$ lb./in.^{2}. *Ans.* 910 lb.

112. Design of Columns by Euler's Formula.—The determination of the size of a column of regular section to carry a given load is a simple matter of substitution in the formula. If material is available only in certain commercial sizes, the next commercial size larger than the computed value must, of course, be selected.

In applying Euler's formula to the selection of a rolled structural shape of the proper size, the value of the necessary moment of inertia is computed and the selection is made from the table of properties of sections the same as in the case of beams. The least value of the moment of inertia governs.

If the value of l/k is below 160 for steel columns or below 100 for wood columns, Euler's formula should not be used. Another computation should be made with a formula from one of the succeeding chapters.

PROBLEMS

1. Compute the diameter of a round-end steel column 20 ft. long to carry a load of 6000 lb. with a factor of safety of 2.25. Use $E = 29,000,000$ lb./in.^{2}. *Ans.* 2.72 in.

2. Solve Problem 1 if the column has fixed ends instead of round ends. *Ans.* 1.924 in.

3. Assuming round-end conditions, compute the size of a square wooden column 16 ft. long to carry a load of 1800 lb. with a factor of safety of 3. Use $E = 1,600,000$ lb./in.^{2}. *Ans.* 3.5 in.

4. A hollow steel column 8 ft. long to carry a load of 600 lb. is to have its inside diameter two-thirds of its outside diameter. Using $E = 30,000,000$ lb./in.^{2} and a factor of safety of 2.5, compute the outside diameter. *Ans.* 1.042 in.

5. Select an I-beam to be used as a column 30 ft. long to carry a load of 20,000 lb. with a factor of safety of 2.25. Use $E = 29,000,000$ lb./in.^{2}, and assume round-end conditions. *Ans.* 18-in. 54.7-lb. I-beam.

6. Select a rolled angle section with equal legs to be used as a column 15 ft. long to carry a load of 6000 lb. with a factor of safety of 2.5. Use $E = 29,000,000$ lb./in.^{2} and assume round-end conditions. *Ans.* 4- by 4- by $\frac{3}{8}$-in. angle.

7. Select a rolled Z-bar section to be used as a column 20 ft. long to carry a load of 4000 lb. with a factor of safety of 2.25. Assume round-end conditions and use $E = 29,000,000$ lb./in.^{2}. *Ans.* 5- by $3\frac{1}{4}$- by $\frac{5}{16}$-in. Z-bar.

CHAPTER XIV

RANKINE'S COLUMN FORMULA

113. Rational Basis for Form of Rankine's Formula.—As stated previously, Euler's formula is not valid for steel columns if their slenderness ratio l/k is less than about 160. Since the value of l/k for columns used in structural work is usually between 60 and 160, it becomes necessary to use a column formula which takes into account the direct component of the stress as well as the flexural component.

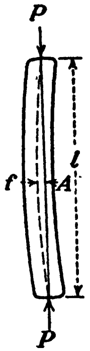


FIG. 165.

Figure 165 represents a column of medium length which is loaded with an axial load P. At the middle point A on the concave side where the deflection is f, the unit stress is

$$S = \frac{P}{A} + \frac{Pfc}{I}.$$

Since

$$I = Ak^2,$$

$$S = \frac{P}{A}\left(1 + \frac{fc}{k^2}\right).$$

The unit load is

$$\frac{P}{A} = \frac{S}{1 + \frac{fc}{k^2}}.$$

The following relations were found to hold true for beams:

$$S = \frac{\alpha Wlc}{I}; \qquad f = \frac{\beta Wl^3}{EI}.$$

In these equations, α and β are constants, depending upon the support and loading of the beam, and f is the maximum deflection. If the quantity W is eliminated between these equations, the following expression is obtained:

$$fc = \frac{\beta S l^2}{\alpha E}.$$

For beams of the same material loaded until the same maximum stress S is reached, $fc = Kl^2$, in which K is a constant and is equal to $\beta S/\alpha E$.

Since columns are of the same form as beams, the only difference being in the method of loading, it is reasonable to assume that in the case of columns, also, fc varies as l^2, or

$$fc = ql^2,$$

in which q is a constant depending upon the material of the column and also upon its method of loading and support. In contrast with beams, no rational method of determining the value of the deflection of columns is available, so the constant q is usually determined by experiment.

If the quantity ql^2 is substituted for fc in the expression for the unit load, it becomes

$$\frac{P}{A} = \frac{S}{1 + q\left(\frac{l}{k}\right)^2}.$$

The formula just derived is Rankine's formula for columns of medium length. If S becomes S_u, the yield point of the material, the formula gives the ultimate unit load on the column. If S is the safe stress for short blocks of the material in compression, the formula gives the safe unit load on the column.

Rankine's formula is sometimes called Gordon's formula, since it is only a modification of the earlier Gordon's formula,

$$\frac{P}{A} = \frac{S}{1 + q\left(\frac{l}{d}\right)^2},$$

in which d is the least dimension of the column.

114. Rankine's and Other Constants.—From the results of a comparatively small number of tests of columns with rather low values of the slenderness ratio, Rankine determined the value of 1/18,000 for q for round-end columns. He used the value of

12,500 for S, so his formula for round-end columns became

$$\frac{P}{A} = \frac{12{,}500}{1 + \frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}.$$

This corresponds to a factor of safety of nearly 3, since the ultimate value of S for medium steel is used as 36,000 lb./in.^{2}. Curve b, Fig. 166, shows Rankine's curve with Rankine's constant.

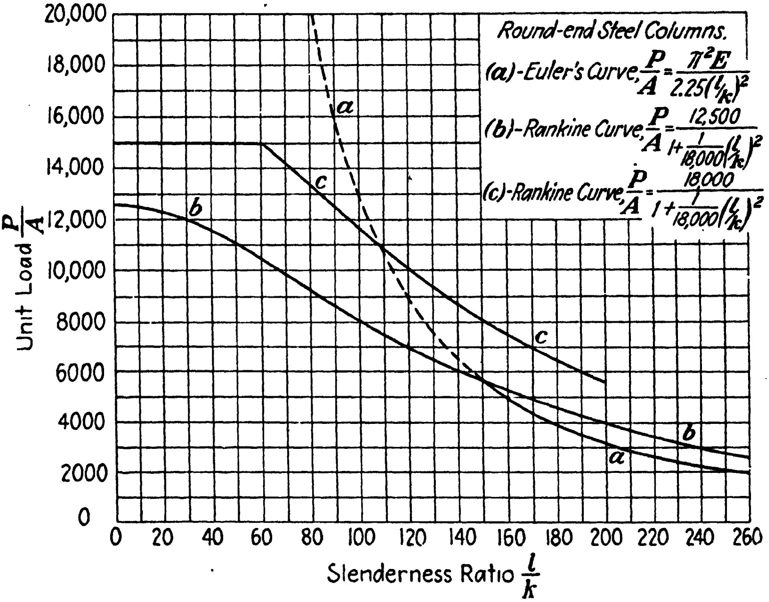


FIG. 166.

This curve gives low values of P/A for values of l/k between 0 and 80.

The value of the constant q determined by Rankine for square-end columns was 1/36,000. If the ends approach fixed-end conditions and there is no eccentricity of loading, these values are safe. Since a slight displacement of a square-end or fixed-end column, either axially or laterally, will cause an unknown eccentricity of loading, it is becoming common practice to treat square and fixed ends as no better than round or pin ends.

For values of l/k between 60 and 200, the American Institute of Steel Construction uses the following Rankine formula:

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}.$$

The factor of safety for columns computed by this formula is only 2. For columns with values of l/k between 0 and 60, no column formula is used, but a maximum value of 15,000 for P/A is specified. This curve is shown in Fig. 166, curve c.

PROBLEM

1. On a diagram similar to Fig. 166, plot the curve $\frac{P}{A} = \frac{15{,}000}{1 + \frac{1}{15{,}000}\left(\frac{l}{k}\right)^2}$.

115. Limits for Use of Rankine's Formula.—It is becoming increasingly common practice for building laws to specify a maximum value of P/A less than 16,000 for medium steel, on account of the unknown eccentricity. The American Institute of Steel Construction, the American Bridge Company and the New York Central Railway, for instance, specify a maximum value of 15,000. The Chicago Bridge and Iron Works specifies a maximum value of 14,000, the Bethlehem Steel Company specifies 13,000 and the American Railway Engineering Association specifies 12,500 for railway bridges. The formula given in Problem 1, Art. 114, may be used safely in the lower ranges of values of l/k. Rankine's formula with Rankine's constants gives values which are too conservative in this range.

In structural work, it is common practice to limit the upper value of l/k. For example, the American Institute of Steel Construction specifies 120 as the upper limit of l/k for main members and 200 as the upper limit for secondary members, such as wind bracing. The Building Code Committee of the U. S. Department of Commerce, in their Report BH9, recommend 160 as the upper limit of l/k in buildings.

In certain kinds of machines and sometimes in temporary building construction, columns with values of l/k in excess of 200 may be used. Rankine's formula with Rankine's constants may be used in this range, as may also the formula given in Problem 1, Art. 114. For columns with high values of l/k, a check with the result from Euler's formula is advisable.

116. Investigation of Columns by Rankine's Formula.—Rankine's formula is easily applied to the investigation of the safe load to be placed upon a given column, or to the determination of the factor of safety with a given loading. If the curve is plotted on a diagram, the safe unit load may be read off directly.

EXAMPLE

A column is composed of a web plate 10 by ½ in. and four angles, each 6 by 4 by ½ in., with the short legs riveted to the plate and the long legs spaced 10.5 in. back to back. By means of the formula of the American Institute of Steel Construction, $\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}$, compute the total safe load on the column for lengths of 10 ft. and 20 ft.

Solution.—The moment of inertia of the cross section with respect to the axis perpendicular to the web plate is 412 in.^{4}. That with respect to the axis parallel to the web plate is 165 in.^{4}. The cross-section area is 24 sq. in. The least radius of gyration is $k = \sqrt{165/24} = 2.62$ in. For a length of 10 ft., $l/k = 120/2.62 = 45.8$. Since this is below 60, the value of P/A is limited to 15,000 lb./in.^{2}.

$$P = 24 \times 15{,}000 = 360{,}000 \text{ lb.}$$

For a length of 20 ft., $l/k = 240/2.62 = 91.6$. For this value of the slenderness ratio, the column formula applies.

$$\frac{P}{24} = \frac{18{,}000.}{1 + \frac{91.6 \times 91.6}{18{,}000}}.$$

$$P = 24 \times 12{,}280 = 294{,}700 \text{ lb.}$$

PROBLEMS

1. By means of the formula of the American Institute of Steel Construction compute the safe load on a 10-in. 25.4-lb. I-beam used as a column for lengths of 4 ft. and 12 ft. *Ans.* 110,700 lb.; 59,700 lb.

2. For a 15-in. 75-lb. I-beam used as a column in accordance with the specifications of the American Institute of Steel Construction, what are the upper limits of the overall length, (1) for which no column formula is used; (2) for main members; (3) for secondary members?
Ans. 5 ft. 10.8 in.; 11 ft. 9.6 in.; 19 ft. 8 in.

3. A solid circular steel column 4 in. in diameter and 4 ft. long carries a load of 180,000 lb. Compute the factor of safety by the specifications of the American Institute of Steel Construction. *Ans.* 2.51.

4. By means of the formula given in Problem 1, Art. 114, compute the safe load on a steel column 6 in. square and 10 ft. long. *Ans.* 409,000 lb.

5. By means of the specifications of the American Institute of Steel Construction, compute the safe load on a column 40 ft. long with a cross section as shown in Fig. 167. *Ans.* 272,000 lb.

6. The American Railway Engineering Association specifies the formula $P/A = 15{,}000 - (1/3)(l/k)^2$ for pin-end columns in railway bridges for values of l/k up to 140. Compute the safe load on the column referred to in Problem 5 by means of this formula. *Ans.* 277,000 lb.

7. By means of the specifications of the American Institute of Steel Construction compute the safe load on a column consisting of a structural angle section, 8 by 6 by 1 in. in cross section and 20 ft. long.

Ans. 79,300 lb. (secondary member only).

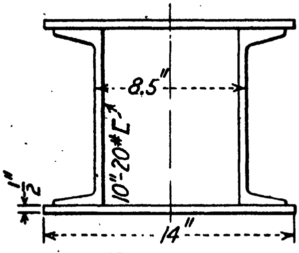


FIG. 167.

117. Design of Columns by Rankine's Formula.—In using Rankine's formula to determine the size of a column of circular, square, or other regular-sided cross section to carry a given load, a quadratic equation in the square of some dimension is obtained. In determining the size of a column built of rolled structural shapes, unless tables computed by the required formula are available, the method of solution by trial is the simplest. Since for main members the value of P/A varies from 15,000 pounds per square inch down to 10,000 pounds per square inch, an approximate value of the area of cross section of a column to carry a load of 600,000 pounds must be between 40 and 60 square inches. A column with a cross-section area within this range should be selected for the first computation. If the size selected is too large or too small, another trial must be made. Tables in the Handbook of the American Institute of Steel Construction and other steel handbooks give safe loads upon single rolled H-beams used as columns.

EXAMPLE 1

By means of the formula $\dfrac{P}{A} = \dfrac{15{,}000}{1 + \dfrac{1}{15{,}000}\left(\dfrac{l}{k}\right)^2}$, compute the size of a solid circular steel column 10 ft. long to carry a load of 180,000 lb.

Solution.—Let r be the radius of the cross section.

$$\frac{180{,}000}{\pi r^2} = \frac{15{,}000}{1 + \dfrac{120 \times 120 \times 4}{15{,}000 \times r^2}}.$$

$$r^4 - 3.82r^2 - 14.66 = 0.$$

$$r^2 = 6.19.$$

$$r = 2.49 \text{ in.}$$
$$\text{Diam.} = 4.98 \text{ in.}$$

This formula is valid for all values of l/k.

EXAMPLE 2

By means of the specifications of the American Institute of Steel Construction, select a plate and angle column 25 ft. long to carry a load of 400,000 lb. The column is to be a main member.

Solution.—The area of the column must be between 400,000/15,000 = 26.7 and 400,000/10,000 = 40 sq. in. For trial, assume an area of 35 sq. in. If a plate 12 by ⅝ in. is used, with four angles, each 8 by 3½ by ⅝ in., with the short legs riveted to the plate and the long legs spaced 12½ in. back to back, the area is 34.7 sq. in. The moment of inertia with respect to the axis normal to the web plate is 927.6 in.^{4}. That with respect to the axis parallel to the web plate is 483.0 in.^{4}, and therefore governs. $k^2 = 13.92$.

$$P = \frac{34.7 \times 18{,}000}{1 + \dfrac{300 \times 300}{18{,}000 \times 13.92}} = 460{,}000 \text{ lb.}$$

Since this value is larger than needed, another trial will be made, with a smaller section. A web plate 10 by 9/16 in. and 4 angles, each 8 by 3½ by 9/16 in., spaced 10½ in. back to back will be used. The area is 30.23 sq. in. The moment of inertia with respect to the axis normal to the web plate is 565 in.^{4}. That with respect to the axis parallel to the web plate is 428 in.^{4}, and therefore governs. $k^2 = 14.16$.

$$P = \frac{30.23 \times 18{,}000}{1 + \dfrac{300 \times 300}{18{,}000 \times 14.16}} = 402{,}000 \text{ lb.}$$

The value of l/k is 300/3.76 = 80. This section is satisfactory.

PROBLEMS

1. By means of the specifications of the American Institute of Steel Construction, compute the size of a square column 6 ft. long to carry a load of 750,000 lb. *Ans.* 7.07 in.

2. Solve Problem 1 for a length of 20 ft. *Ans.* 8.1 in.

3. Solve Problem 1 for a length of 30 ft.

Ans. 9.18 in. if used as a secondary member; 10.392 in. if used as a main member.

4. By means of the formula $\dfrac{P}{A} = \dfrac{15{,}000}{1 + \dfrac{1}{15{,}000}\left(\dfrac{l}{k}\right)^2}$, determine the size of a plate and channel column 30 ft. long to carry a load of 600,000 lb.

Ans. Two 15-in. 40-lb. channels spaced 8 in. back to back, and two 16- by 1-in. plates.

CHAPTER XV

STRAIGHT-LINE COLUMN FORMULA

118. Graphic Derivation of Straight-line Column Formula.—It will be seen by reference to Fig. 166 that between the values of $l/k = 60$ and $l/k = 160$, the Rankine curves do not vary much from a straight line. Since the great majority of columns used in structural work fall within this range, the equations of such straight lines may be used as column formulas. Although not quite so accurate as the Rankine formulas and more limited in range, the straight-line formulas have the advantage of simplicity of form and application. Although not used so much as formerly, they still have a place in American engineering.

As given in analytical geometry, the equation of a straight line is

$$y = mx + b.$$

In this equation, b is the intercept on the Y axis and m is the slope of the line. In the column diagram, the ordinate y is P/A and the abscissa x is l/k. The graphic method of deriving the equation of a safe straight-line formula consists in plotting Euler's curve on a diagram, drawing a straight line through the allowable value of P/A on the Y axis and tangent to Euler's curve, and then determining the equation of this straight line. Let the allowable value of P/A on the Y axis be called S. The equation of the straight line then becomes

$$\frac{P}{A} = S + m\frac{l}{k}.$$

If the value of S is taken as 18,000 and the value of the yield point as 36,000, the factor of safety is 2. In Fig. 168, Euler's curve is plotted with this factor of safety. Through the point $S = 18{,}000$, the straight line SBC is drawn, tangent to the curve at B and intersecting the X axis at the point $l/k = 231$. The value of m is $18{,}000/231 = 78$, and is negative in sign, since the line slopes downward to the right. The equation then becomes

$$\frac{P}{A} = 18,000 - 78\frac{l}{k}.$$

If the value of S is taken as 16,000 and the value of the yield point as 36,000, the factor of safety is 2.25. In Fig. 169, Euler's curve is plotted with 2.25 as the factor of safety. Through the point S = 16 000, the straight line SBC is drawn, tangent to the Euler curve at B and intersecting the X axis at the point where l/k is 231. The equation of this straight line then becomes

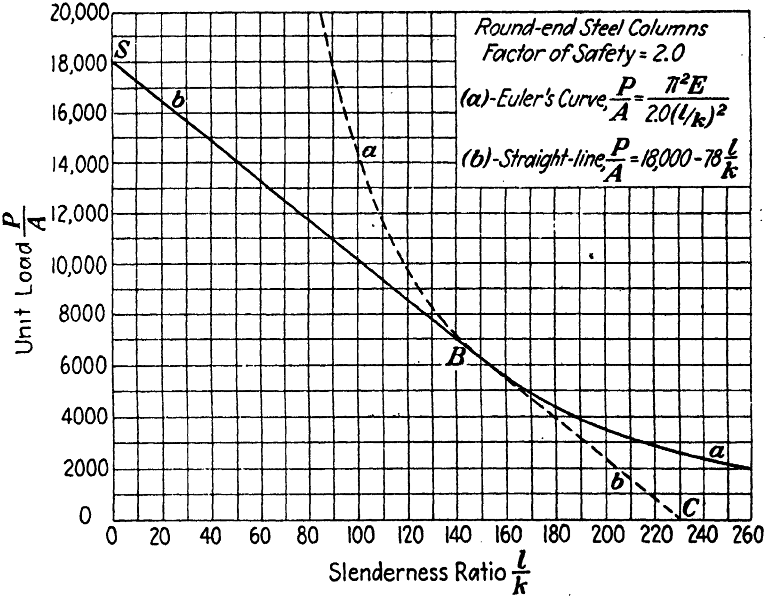


FIG. 168.

$$\frac{P}{A} = 16,000 - 69.3\frac{l}{k}.$$

This formula is commonly used as

$$\frac{P}{A} = 16,000 - 70\frac{l}{k}.$$

Some straight-line formulas have been made arbitrarily, with no direct relation to any Euler curve. For example, the Building

Code Committee of the Department of Commerce in their Report BH9, 1926, recommend the formula

$$\frac{P}{A} = 18{,}000 - 70\frac{l}{k}$$

for columns made from steel which passes the specifications of the American Society for Testing Materials (see Table XV).

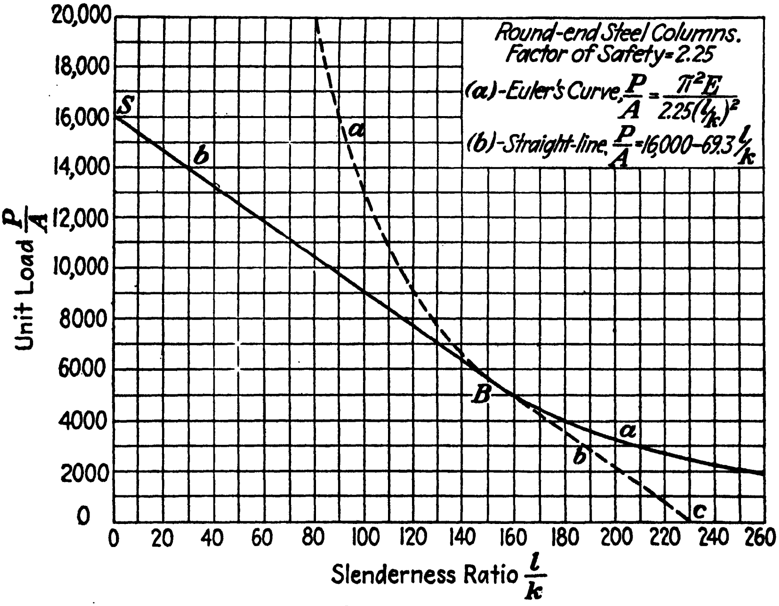


FIG. 169.

The Euler curve to which this straight line is tangent has a factor of safety of only 1.63 while the value of S has a factor of safety of 2.

PROBLEMS

1. On a diagram similar to Fig. 168, plot Euler's curve corresponding to $S = 15{,}000$ and derive the equation of the straight-line curve tangent to it.

Ans. $\frac{P}{A} = 15{,}000 - 65\frac{l}{k}$.

2. For steel which is considered acceptable, but of which the origin and physical characteristics are not definitely determined, the Building Code Committee referred to above recommends the column formula, $P/A = 16{,}000 - 60l/k$. Determine the factor of safety of the Euler curve to which it is tangent. *Ans.* 1.7.

119. Algebraic Derivation of Straight-line Column Formula.—In terms of x and y, Euler's formula becomes

$$y = \frac{\pi^2 E}{x^2}.$$

If this expression is differentiated with respect to x, it gives the slope of Euler's curve at any point.

$$\frac{dy}{dx} = -\frac{2\pi^2 E}{x^3}.$$

At the point B, Fig. 170, where the straight line SBC is tangent to Euler's curve, the slope m of the straight line is the same as the slope of Euler's curve.

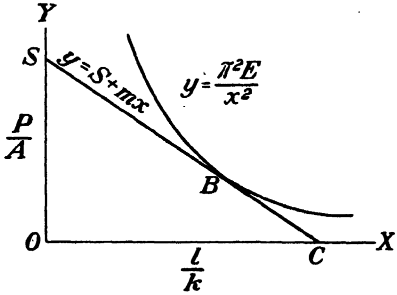


FIG. 170.

If x_B is the abscissa of point B, the slope of Euler's curve at B is,

$$m = -\frac{2\pi^2 E}{{x_B}^3}.$$

The equation of the straight line then becomes,

$$y = S - \frac{2\pi^2 E}{{x_B}^3}x.$$

At point B, the coordinates (y_B, x_B) must satisfy both the straight-line equation and Euler's equation.

$$y_B = S - \frac{2\pi^2 E}{{x_B}^2}.$$

$$y_B = \frac{\pi^2 E}{{x_B}^2}.$$

If the quantity π^2E/x_B^2 is eliminated between these two equations, the result becomes,

$$y_B = \frac{S}{3}.$$

If a straight line is drawn through any point S on the Y axis and tangent to Euler's curve, the ordinate of the point of tangency is one-third of S.

The relation just derived is true for any value of S and for Euler's curve with any factor of safety, but in order to have a consistent column formula, the factor of safety of S and the Euler curve should be the same.

If the value of S is 16,000, y_B becomes 16,000/3 or 5,333. If this value is substituted in Euler's equation, $y = \pi^2E/2.25x^2$, it becomes,

$$5{,}333 = \frac{9.87 \times 29{,}000{,}000}{2.25x_B^2}.$$

$$x_B^2 = 23{,}800.$$

$$x_B = 154.$$

The slope of the straight-line curve is given by the expression,

$$m = -\frac{S - y_B}{x_B} = -\frac{10{,}667}{154} = -69.3.$$

The straight-line equation then becomes,

$$\frac{P}{A} = 16{,}000 - 69.3\frac{l}{k}.$$

PROBLEMS

1. Check the equation $P/A = 18{,}000 - 78l/k$ by the algebraic method.

2. Derive the straight-line formula with a factor of safety of 2.5.

Ans. $\frac{P}{A} = 14{,}400 - 62.4\frac{l}{k}.$

3. For steel with a yield point of 36,000 lb./in.^{2}, the American Bridge Company used the formula $P/A = 20{,}000 - 70l/k$ in the design of the Southern Pacific Railroad bridge across Suisun Bay near San Francisco. Determine the factor of safety of the Euler curve to which it is tangent.

Ans. 1.18.

120. Limits for the Use of Straight-line Formulas.—Due to the unknown effect of unavoidable eccentric loading on columns which have a small slenderness ratio, it is quite general practice to specify as a maximum unit load, a value less than the value of S in the column formula. The Chicago Building Laws specify the formula

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{k},$$

for values of l/k between 30 and 150. For values of l/k below 30, a maximum value of 14,000 for P/A is specified.

If a straight-line curve is tangent to Euler's curve, it should not be used for values of l/k much higher than the point of tangency. That is, if the computed value of P/A is less than $S/3$, the result should not be used, but another computation should be made with Euler's curve or with some Rankine curve which is valid for that value of l/k.

If a straight-line curve has no relation to Euler's curve, the two should be plotted on the same diagram to determine the range of values for which the formula may be considered valid.

All of the straight-line formulas have been derived for round-end or pin-end conditions, but, as was stated in Art. 114, it is becoming common practice to consider square-end and fixed-end columns as no stronger than pin-end columns on account of the unknown eccentric loading. The straight-line formulas are used, therefore, with columns of any end conditions.

PROBLEMS

1. By means of the formula $P/A = 16{,}000 - 70l/k$, compute the allowable load on a steel column 4 in. in diameter and 20 ft. long. Explain the result.

2. Estimate the limits of the value of l/k for which the formula $P/A = 20{,}000 - 70l/k$ should be used for columns made of medium structural steel with a yield point of 36,000 lb./in.^{2}.

121. Investigation of Columns by Straight-line Formulas.—

The investigation of a column by means of any straight-line formula is a very simple computation. If the largest value of the slenderness ratio l/k for the column is substituted in the formula, the allowable unit load P/A is obtained. If the curve is plotted on a diagram, the value of the unit load may be read off directly from the curve for the given value of l/k. The product of the unit load and the area of the cross section gives the total safe load on the column.

PROBLEMS

1. Compute the total safe load on a steel column 4 in. in diameter and 8 ft. long by means of the formula $P/A = 16{,}000 - 70l/k$.

Ans. 116,700 lb.

2. Compute the safe load on a 12-in. 31.8-lb. I-beam 12 ft. long used as a column, by means of the formula $P/A = 18{,}000 - 78l/k$.

Ans. 63,700 lb.

3. Compute the safe load on an angle section 6 by 6 by 1 in. and 10 ft. long, used as a column, by means of the formula $P/A = 16{,}000 - 70l/k$.

Ans. 96,360 lb.

4. The cross-section area of a 16-in. Bethlehem H-column with two 20- by 3-in. flange plates is 226.86 sq. in., and the least radius of gyration is 5.11 in. By means of the formula $P/A = 18{,}000 - 70l/k$, compute the total safe load on this column for lengths of 20 ft. and 40 ft.

Ans. 3,338,000 lb.; 2,592,000 lb.

5. A column 40 ft. long is composed of two 15-in. 45-lb. channels spaced 9.764 in. back to back, and two cover plates, each 18 by 2 in. Compute the total safe load by means of the formula $P/A = 30{,}000 - 105l/k$.

Ans. 2,026,700 lb.

122. Design of Columns by Straight-line Formulas.—In using a straight-line formula to determine the size of a column of circular, square, or other regular-sided cross section to carry a given load, a simple quadratic in some dimension of the cross section is obtained.

As in the case of Rankine's formula, in determining the size of a column built of rolled structural shapes, the method of trial is the simplest. A column is selected by estimation, or, if available, from a table of safe loads computed by any other column formula. Computation by the required formula will show whether the column selected is the proper size or not.

EXAMPLE 1

By means of the formula $P/A = 16{,}000 - 70l/k$, compute the size of a square steel column 15 ft. long to carry a load of 400,000 lb.

Solution.—Let b represent the length of one side of the cross section.

$$\frac{400{,}000}{b^2} = 16{,}000 - 70\frac{180 \times 3.464}{b}.$$
$$4000 = 160b^2 - 436.5b.$$
$$b = 6.54 \text{ in.}$$

The radius of gyration $k = 6.54/3.464 = 1.89$ in. The slenderness ratio $l/k = 180/1.89 = 95$. This value is within the limits for which this formula is valid.

EXAMPLE 2

By means of the formula $P/A = 18{,}000 - 70l/k$, select an H-column 40 ft. long to carry a load of 1,230,000 lb.

Solution.—In the Handbook of the American Institute of Steel Construction, the lightest H-column which will carry this load is one weighing 384 lb./ft. It has a cross-section area of 112.93 sq. in., its least radius of gyration is 4.29 in., and it will carry 1,234,C00 lb., computed by the formula of the American Institute of Steel Construction. By the given formula,

$$P = 112.93\left(18{,}000 - 70 \times \frac{480}{4.29}\right) = 1{,}147{,}000 \text{ lb.}$$

This amount lacks 83,000 lb., so the next selection will be a column which will carry at least 1,234,000 + 83,000 = 1,317,000 lb. This is one weighing 412 lb. per linear ft. It has a cross section of 121.15 sq. in., its least radius of gyration is 4.32 in., and it will carry 1,334,000 lb. by the formula of the American Institute of Steel Construction. By the required formula,

$$P = 121.15\left(18{,}000 - 70\frac{480}{4.32}\right) = 1{,}238{,}500 \text{ lb.}$$

This column is satisfactory.

PROBLEMS

1. By means of the formula $P/A = 16{,}000 - 70l/k$, compute the diameter of a solid circular steel column 10 ft. long to carry a load of 200,000 lb.
Ans. 5.17 in.

2. If the column to carry the load given in Problem 1 is to be hollow, with the inside diameter three-fourths of the outside diameter, how much lighter is it than the solid column? *Ans.* 154 lb.

3. Select a Carnegie-beam section 32 ft. long to be used as a column to carry a load of 500,000 lb. in accordance with the formula $P/A = 15{,}000 - 65l/k$. *Ans.* 14-in. 193-lb. beam.

4. By means of the formula $P/A = 18{,}000 - 78l/k$, determine the size of a single rolled-angle column 8 ft. long to carry a load of 60,000 lb.
Ans. 6- by 6- by ½-in. angle.

5. By means of the formula $P/A = 18{,}000 - 70l/k$, select a column 40 ft. long composed of two channels and one cover plate to carry a load of 750,000 lb.

Ans. Two 15-in. 45-lb. channels, 12 in. back to back, and one 20- by $1\frac{5}{16}$-in. plate.

6. By means of the formula $P/A = 18{,}000 - 78l/k$, select a wide flange H-column 50 ft. long with two flange plates to carry a load of 2,000,000 lb.

Ans. 14-in. 320-lb. core section with two plates each 22 by 2¾ in.

CHAPTER XVI

COLUMNS IN GENERAL

123. Timber Columns.—Prior to 1927, several technical organizations, including the American Society for Testing Materials and the American Railway Engineering Association, cooperated with the Forest Products Laboratory of the U. S. Forest Service in a series of comprehensive tests upon timber to determine its properties and safe working stresses. In accordance with the results of their tests, these organizations recommend a fourth-power parabola for columns of intermediate length. This curve passes through point S on the P/A axis and is very flat for low values of the slenderness ratio, but curves sharply to become tangent to Euler's curve at a point for which P/A is two-thirds of S. The formula is given in terms of the least dimension d of the column:

$$\frac{P}{A} = S\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right].$$

The quantity S is the allowable compressive stress for short blocks parallel to the grain. The quantity K is a constant for any grade and species of timber, and is the value of the quantity l/d at the point where the parabola is tangent to Euler's curve (see Table V for values of S and K). For any grade and species of timber,

$$K = \frac{\pi}{2}\sqrt{\frac{E}{6S}}.$$

The Forest Products Laboratory formula has been adopted by the Building Code Committee of the U. S. Department of Commerce and by the National Lumber Manufacturer's Association, so it is the most authoritative timber column formula in use at present.

Commercial timber is divided into two grades, select and common, the allowable stresses for the common grade being

80 per cent of the values for the select grade for most species. Three conditions of exposure are recognized; (1) dry, as in ordinary buildings; (2) occasionally wet but soon dried, as in railway trestles, bridges, and towers; (3) continually damp or wet, as in packing plants, docks, and posts in soil. For conditions (2) and (3), the allowable stresses are reduced by amounts varying from 12½ to 33⅓ per cent, depending upon the species of timber and the degree of exposure.

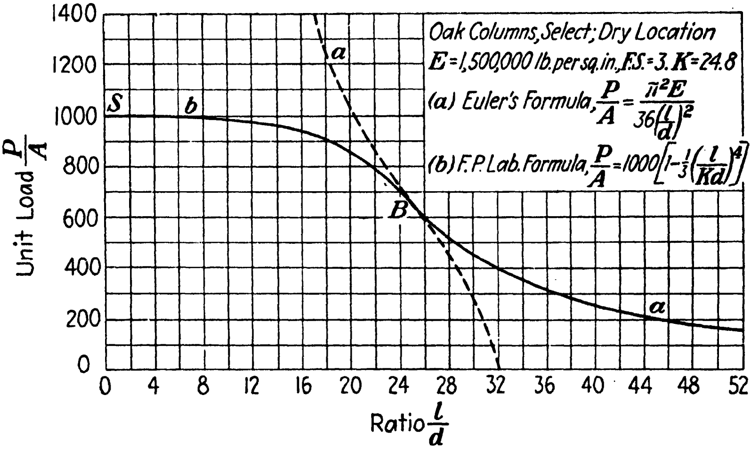


FIG. 171.

In Fig. 171 is shown Euler's curve for select oak columns in dry locations, with a factor of safety of 3. Its equation is

$$\frac{P}{A} = \frac{\pi^2 E}{36\left(\frac{l}{d}\right)^2}.$$

Curve b is the Forest Products Laboratory equation for oak columns, drawn through the point $S = 1000$ and tangent to Euler's curve at point B where $P/A = 667$ and l/d is about 25. The Forest Products Laboratory formula is to be used for values of l/d between 0 and 25; Euler's formula is to be used for values between 25 and 50. Values of l/d greater than 50 are not permitted.

In terms of the least radius of gyration, the Forest Products Laboratory formula for rectangular oak columns becomes

$$\frac{P}{A} = 1000\left[1 - \frac{1}{164,000,000}\left(\frac{l}{k}\right)^4\right].$$

This curve is shown plotted from point S to point B, Fig. 172. Curve a is Euler's curve in terms of the least radius of gyration k, with a factor of safety of 3. Its equation is

$$\frac{P}{A} = \frac{\pi^2 E}{3\left(\frac{l}{k}\right)^2}.$$

For west-coast Douglas fir and for southern pine, the value of E is 1,600,000 pounds per square inch and the formula for rectangular columns of select timber in dry locations becomes

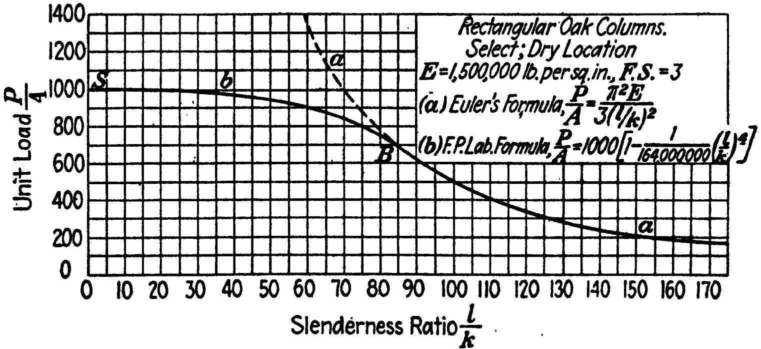


FIG. 172.

$$\frac{P}{A} = 1175\left[1 - \frac{1}{136,000,000}\left(\frac{l}{k}\right)^4\right].$$

For west-coast hemlock, the value of E is 1,400,000 and the formula for rectangular columns becomes

$$\frac{P}{A} = 900\left[1 - \frac{1}{177,000,000}\left(\frac{l}{k}\right)^4\right].$$

For spruce and Rocky Mountain Douglas fir, the value of E is 1,200,000 and the formula for rectangular columns is

$$\frac{P}{A} = 800\left[1 - \frac{1}{164,000,000}\left(\frac{l}{k}\right)^4\right].$$

The formulas for other species of timber may be computed from the values given in Table V. For complete data for modification of these formulas for grade of timber and condition of exposure, see the "Book of Standards," 1927. Part II, of the American Society for Testing Materials.

PROBLEMS

1. Compute the size of a square southern-pine column 16 ft. long to carry a load of 300,000 lb. *Ans.* 16.1 in.

2. Compute the total safe load on an oak column 6 by 10 in. in cross section, for lengths of 10 ft. and 20 ft. *Ans.* 51,500 lb.; 15,400 lb.

3. Derive the Forest Products Laboratory formula in terms of the slenderness ratio l/k for redwood timber columns of rectangular cross section, select grade, in dry locations. Use $E = 1{,}200{,}000$ lb./in.^{2}.

Ans. $\frac{P}{A} = 1000\left[1 - \frac{1}{105{,}200{,}000}\left(\frac{l}{k}\right)^4\right].$

4. Compute the size of a square redwood column 30 ft. long to carry a load of 250,000 lb. *Ans.* 17.9 in.

5. Solve Problem 4 for a length of 60 ft. *Ans.* 25.1 in.

6. Derive the Forest Products Laboratory formula in terms of the slenderness ratio l/k for spruce columns of circular cross section, select grade, in dry locations. *Ans.* $\frac{P}{A} = 800\left[1 - \frac{1}{292{,}250{,}000}\left(\frac{l}{k}\right)^4\right].$

124. Cast-iron Columns.—Euler's column formula applies to cast-iron columns for which the slenderness ratio is large, but on account of the brittle nature of ordinary cast-iron, such slender columns are rarely used. For cast-iron columns with values of the slenderness ratio l/k below 100, the following Rankine formula has been used.

$$\frac{P}{A} = \frac{10{,}000}{1 + \frac{1}{12{,}800}\left(\frac{l}{k}\right)^2}.$$

The Building Code Committee of the U. S. Department of Commerce in their Report BH9, 1926, for values of l/k below 90, recommend the straight-line formula,

$$\frac{P}{A} = 9000 - 40\frac{l}{k}.$$

For values of l/k between 90 and 120, they recommend the formula,

$$\frac{P}{A} = 6000 - 27\frac{l}{k}.$$

PROBLEMS

1. By means of the Rankine formula given above, compute the safe load on a cast-iron column 12 in. outside diameter, 10 in. inside diameter, and 20 ft. long. *Ans.* 267,000 lb.

2. By means of the Rankine formula given above, compute the size of a square cast-iron column 8 ft. long to carry a load of 80,000 lb. *Ans.* 3.64 in.

3. By means of the straight-line formula given above, compute the safe load on a cast-iron column 15 in. in diameter and 18 ft. long. *Ans.* 1,183,000 lb.

4. A hollow circular cast-iron column 25 ft. long to carry a load of 1,600,000 lb. is to have the inside diameter four-fifths of the outside diameter. By means of the straight-line formula given above, compute the outside diameter. *Ans.* 27.28 in.

125. Concrete Columns.—Concrete columns are not designed by a column formula, but a maximum value of the ratio of the length l to the least lateral dimension d is usually specified. The maximum allowable value of l/d is usually 15. For a circular column, this corresponds to a value of 60 for the slenderness ratio. For a square column, it corresponds to a value of 52.

For plain concrete columns, the building codes of many cities also specify an allowable value of the unit stress in the concrete lower than the value of 650 pounds per square inch, the value which is commonly allowed in bearing and in flexure.

For concrete columns which are reinforced with longitudinal bars, the code of the American Society of Civil Engineers permits the following increase of load. If P is the total load, S_c the allowable compressive stress in the concrete, A_s the area of the steel, and n the ratio of the moduli of elasticity, the allowable load P is given by the expression,

$$P = S_c(A_c + nA_s).$$

For columns reinforced with hoop or spiral reinforcement only, a higher value of S_c than that for plain concrete is commonly permitted.

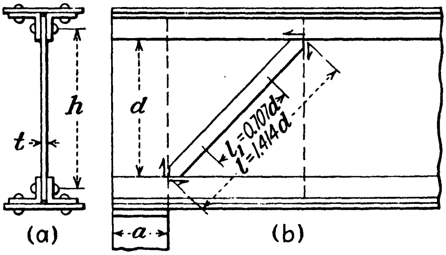
PROBLEMS

1. Circular concrete columns 10 ft. high and spaced 16 ft. apart each way, are to support a floor load of 600 lb./ft.^{2} of floor area, with an allowable value of S_c = 650 lb./in.^{2}. Compute the size of the columns. *Ans.* 17.4 in.

2. A square concrete column 25 ft. high is to support a load of 220,000 lb. with an allowable value of $S_c = 650$ lb./in.^{2}. Compute the size of the column. *Ans.* 20 in.

3. A concrete column 20 in. in diameter and 15 ft. high is to be reinforced with longitudinal steel rods each 1 in. square. If the allowable value of S_c is 600 lb./in.^{2}, and $n = 15$, how many steel rods will be needed so that the column will carry a load of 280,000 lb.? *Ans.* 11.

126. Column Action in Webs of I-beams and Girders.—It was shown in Art. 18 that in a beam which is subjected to vertical shearing stresses, the direct vertical shear and the induced horizontal shear together develop diagonal compressive and tensile stresses of equal unit intensity with the shear. These stresses act in directions at 45° with the planes of the shear and at 90° with each other. In plate girders and rolled beams, these tensile stresses tend to keep the webs straight, but the compressive stresses tend to buckle them.



(a) (b)

FIG. 173.

Figure 173(*a*) represents an end view and Fig. 173(*b*) a partial side view of a plate girder. If V is the total vertical shear at any section, t the thickness of the web, and h its depth, the unit diagonal compressive stress is

$$S_c = S_s = \frac{V}{th}.$$

If d is the clear distance between the flange angles, the total length of the fixed-end diagonal column upon which the unit load S_c is acting is $l = 1.414d$, or the length of the sine curve is $l_1 = 0.707d$.

If stiffeners are not provided and if the ratio d/t exceeds certain limits, the allowable value of S_c must be reduced in accordance with some column formula. If it is desired to develop the full amount of the allowable shearing stress, stiffeners must be provided at proper intervals. Stiffeners usually consist of pairs of vertical angles riveted to the web of the beam or girder. In some cases the stiffener angles have been replaced by plates welded to the web with their longer axes normal to the web.

The column formula used by the American Institute of Steel Construction is a Rankine formula in terms of d and t, and is derived as follows. Since experiments have shown that until the ratio d/t exceeds 60 there is no danger of column buckling in the web, the formula has the general form, $\frac{P}{A} = \frac{18{,}000}{1 + K\left(\frac{d}{t}\right)^2}$, and passes through the two points (0, 18,000) and (60, 12,000) in the P/A, d/t diagram. The value of 12,000 is the allowable unit stress in shear. If the value of 12,000 is substituted for P/A and 60 for d/t, the formula becomes

$$12{,}000 = \frac{18{,}000}{1 + K(3600)}.$$

$$K = \frac{1}{7200}.$$

With this value of K in the formula above, it becomes

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{7200}\left(\frac{d}{t}\right)^2},$$

as given in Table XVI. In the web of the beam the load P is the vertical shear V, so the formula may be written,

$$\frac{V}{A} = \frac{18{,}000}{1 + \frac{1}{7200}\left(\frac{d}{t}\right)^2}.$$

If this expression is solved for d, it becomes

$$d = 85t\sqrt{18{,}000\left(\frac{A}{V}\right) - 1}.$$

The quantity d in this equation is the maximum allowable clear distance between flanges if no stiffeners are used. If the clear distance between flanges is made to exceed this value of d and stiffeners are used, this equation gives the maximum allowable distance between stiffeners, but in no case should this distance exceed 6 feet.

For the same reduction, the column formula recommended by the Building Code Committee, U. S. Department of Commerce, for steel passing the specifications of the American Society for Testing Materials is a straight-line formula,

$$\frac{V}{A} = 15{,}000 - 70\frac{d}{t}.$$

See Table XV. The maximum allowable value of V/A is 12,000, so this formula is used for values of d/t above 43.

Stiffeners are usually required on the webs of plate girders and rolled beams at all points where there are concentrated loads or reactions and must be of sufficient size to transmit the load or reaction to the web. For all standard rolled beams, the ratio d/t is less than 60, so there is no danger of diagonal column failure due to the compression resulting from the direct and induced shear. There is direct vertical compression in the web above the reaction, however, and if no stiffeners are provided the unit load in the web must be reduced by the column formula

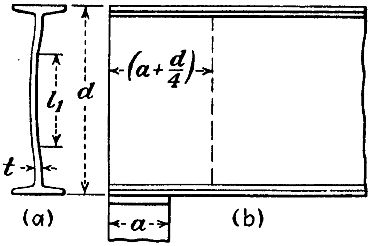


FIG. 174.

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}.$$

The same requirement holds true for plate girders with no end stiffeners.

The web of the beam is considered as a vertical column with fixed ends, so that l_1 of the column formula becomes $d/2$, as shown in Fig. 174. Since $k^2 = t^2/12$, this formula in terms of d and t becomes

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{6000}\left(\frac{d}{t}\right)^2}.$$

The maximum allowable value of P/A is 15,000.

In accordance with the results of tests, the length of web resisting buckling is assumed to be the length of bearing a on the wall plate plus one-fourth of the depth of the beam. The cross-section area of the column is then

$$A = t\left(a + \frac{d}{4}\right).$$

EXAMPLE 1

A plate girder consists of a web plate 48 by ⅜ in. and four flange angles each 6 by 6 by ½ in., spaced 48.5 in. back to back, with the rivets 3½ in. from the back. Using the formula of the American Institute of Steel Construction, compute the maximum allowable end reaction both without stiffener angles and with stiffener angles. Compute the maximum allowable distance between stiffeners at the ends of the girder.

Solution.—The distance between rivet lines is 41.5 in., so the effective area of the web is 41.5 × 0.375 = 15.56 sq. in. The unsupported height of the web between flange angles is 36.5 in. If no stiffener angles are used at the end, the average unit shearing stress S_s must not exceed V/A of the column formula.

$$\frac{V}{A} = \frac{18{,}000}{1 + \dfrac{36.5 \times 36.5}{7200 \times 0.375 \times 0.375}}.$$

$$\frac{V}{A} = 7770 \text{ lb./in.}^2.$$

$$V = 15.56 \times 7770 = 121{,}000 \text{ lb.}$$

If stiffener angles are used, the allowable average unit shearing stress in the web is 12,000 lb./in.².

$$V = 15.56 \times 12{,}000 = 186{,}700 \text{ lb.}$$

The maximum distance between stiffeners at the end of the girder is

$$d = 85 \times 0.375\sqrt{\frac{18{,}000}{12{,}000} - 1} = 22.5 \text{ in.}$$

EXAMPLE 2

A 24-in. 79.9-lb. I-beam rests on a 3.5-in. bearing plate at each end. Compute the maximum allowable end reaction if no stiffeners are used. Compute the maximum web shear V. Compute the length of bearing to develop the maximum end reaction V.

Solution.—For the unsupported web,

$$S_c = \frac{P}{A} = \frac{18{,}000}{1 + \dfrac{24 \times 24}{6000 \times 0.5 \times 0.5}} = 13{,}010 \text{ lb./in.}^2.$$

$$A = t\left(a + \frac{d}{4}\right) = 0.5 \times 9.5 = 4.75 \text{ sq. in.}$$

$$R = S_cA = 13{,}010 \times 4.75 = 61{,}800 \text{ lb.}$$

The maximum web shear is

$$V = 24 \times 0.5 \times 12{,}000 = 144{,}000 \text{ lb.}$$

The length of bearing a to develop an end reaction of 144,000 lb. is given by

$$13{,}010 \times 0.5(a + 6) = 144{,}000.$$
$$a = 16.2 \text{ in.}$$

Instead of an increased length of bearing, stiffener angles could be used in order to develop the full amount of the shear V.

PROBLEMS

1. A plate girder consists of a web plate 72 by ½ in., four flange angles 8 by 6 by 1 in., and two flange plates 18 by 1 in. The 6-in. legs are riveted to the web with one row of rivets 3.5 in. from the backs of the angles, and the 8-in. legs are spaced 72.5 in. back to back. Using the formula of the American Institute of Steel Construction, compute the maximum allowable end reaction with no stiffener angles. If the allowable unit stress in flexure is 18,000 lb./in.^{2}, compute the minimum length in which the full flexural stress may be developed by a load at the middle without causing excessive buckling stresses in the web. Rivet holes are drilled $^{15}\!/_{16}$ in. in diameter.
Ans. 194,000 lb.; 46.4 ft.

2. Solve Problem 1 if stiffener angles are used.
Ans. 393,000 lb.; 22.9 ft.

3. A 20-in. 65.4-lb. I-beam rests on a 3½-in. bearing plate at each end. Compute the maximum allowable end reaction if no stiffener angles are used. If stiffener angles are used so that the total amount of the end shear V may be developed, compute the minimum length in which the full flexural stress of 18,000 lb./in.^{2} may be developed by a uniformly distributed load without causing excessive buckling stresses in the web.
Ans. 60,400 lb.; 5.85 ft.

4. By means of the straight-line formula of the Building Code Committee, solve for the maximum end reaction of the girder described in Problem 1, if no stiffener angles are used. If the allowable unit stress in flexure is 18,000 lb./in.^{2}, compute the minimum length in which the full flexural stress may be developed by a uniformly distributed load without causing excessive buckling stresses in the web. *Ans.* 214,000 lb.; 84 ft.

127. Column Action in Compression Flanges of Beams.—The compression flange of an I-beam, channel beam, or plate girder is subjected to axial compression, and acts to some extent as a column. The action differs from true column action in the fact that at any section the stress varies from a maximum value at the outer fiber to a lesser value near the web, and also in the fact that except in beams of uniform moment, the stress varies from zero at one section to a maximum value at another along the flange.

On account of these variations from true column action, slightly larger values of the unit load P/A are permitted. If beams are unsupported laterally, the allowable unit stress must be reduced according to some column formula. If it is desired that the full value of the compressive stress equal to the allowable tensile stress be developed, lateral supports must be provided at proper intervals.

The column formula specified by the American Institute of Steel Construction is a Rankine formula,

$$\frac{P}{A} = \frac{20{,}000}{1 + \frac{1}{2000}\left(\frac{l}{b}\right)^2} \text{ (see Table XVI).}$$

The length l is the unsupported length and b is the width of the compression flange. The derivation of this formula is similar to that of the formula for diagonal compression in the web of a beam, as given in Art. 126. The value of the allowable compressive stress is 18,000 and the maximum value of l/b for which no reduction is necessary has been found by practice over many years to be about 15. On account of the partial column action, the larger value of 20,000 is used in the numerator. This equation is the equation of the Rankine curve passing through the two points (0, 20,000) and (15, 18,000) in the P/A, l/b diagram. Values of l/b in excess of 40 are not allowed.

For steel passing the specifications of the American Society for Testing Materials, the Building Code Committee of the U. S. Department of Commerce recommends a straight-line formula for the same reduction between the same limits of l/b.

$$\frac{P}{A} = 22{,}000 - 270\frac{l}{b} \text{ (see Table XV).}$$

If $l/b = 15$, the value of P/A is 18,000.

EXAMPLE

By means of the formula of the American Institute of Steel Construction, compute the allowable flexural stress in the compression flange of an 18-in. 54.7-lb. I-beam 20 ft. long supported at the ends and loaded with a uniformly distributed load, with no lateral support. Compute the number of lateral supports if a compressive stress of 18,000 lb./in.^{2} is to be developed.

Solution.—$l = 240$; $b = 6$. **With no lateral supports, the allowable unit stress is**

$$S_c = \frac{P}{A} = \frac{20{,}000}{1 + \dfrac{240 \times 240}{2000 \times 36}} = 11{,}100 \text{ lb./in.}^2.$$

If the beam is to be supported laterally so that a stress of 18,000 lb./in.^{2} may be developed, the length l must not exceed $15b$, or 90 in. For symmetry, there should be four supports, one at each end and one at a point 80 in. from each end.

PROBLEMS

1. By means of the formula of the American Institute of Steel Construction, compute the allowable flexural stress in the compression flange of a 12-in. 31.8-lb. I-beam 20 ft. long supported at the ends and loaded with a uniformly distributed load, if it has no lateral support. Compute also the number of lateral supports if the full compressive stress of 18,000 lb./in.^{2} is to be developed. *Ans.* 9300 lb./in.^{2}; five.

2. Solve Problem 1 by means of the straight-line formula of the Building Code Committee. *Ans.* 9040 lb./in.^{2}; five.

3. A 6-in. 12.5-lb. I-beam 24 ft. long, supported at the ends and loaded with a uniformly distributed load has lateral supports at the ends and at each 6-ft. point along the beam. Compute the allowable stress in compression and the total allowable load on the beam, including its own weight, by means of the formula of the American Institute of Steel Construction. *Ans.* 16,200 lb./in.^{2}; 3285 lb.

128. Eccentric Loads on Columns.

If the load on a column is eccentric, the allowable unit load as determined by any column formula must be still further reduced by the amount Wec/I. W is the amount of the eccentric load and e is the amount of its eccentricity. Figure 175 represents a common method of applying load to a column by means of a bracket. The allowable unit load P/A must equal or exceed the value $(W_1 + W)/A + Wec/I$. W_1 is the direct axial load, W is the eccentric load, and A is the area of the cross section of the column.

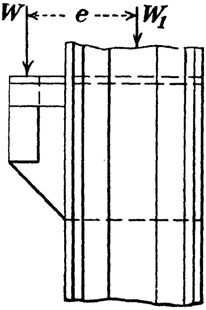


FIG. 175.

EXAMPLE

A column of southern pine of select grade is 6 in. square and 12 ft. long, and carries an axial load of 8000 lb. By means of the formula given in Art. 123, compute the additional load, eccentric 4 in., which can be applied to the column.

Solution:

$$\frac{P}{A} = 1175\left(1 - \frac{144^4}{136{,}000{,}000 \times 9}\right) = 762.$$

$$762 = \frac{8000 + W}{36} + \frac{W \times 4 \times 3 \times 12}{36 \times 36}.$$

$$W = 3890 \text{ lb.}$$

PROBLEMS

1. With an axial load of 2000 lb., compute the additional load eccentric 3 in. which can be applied to the column described in the example above. *Ans.* 6360 lb.

2. The column shown in Fig. 175 is 20 ft. long and consists of a web plate 14 by ⅜ in., four angles each 6 by 4 by 7⁄16 in., and two flange plates each 14 by ⅜ in. The 4-in. legs of the angles are riveted to the web plate and the 6-in. legs are spaced 14.5 in. back to back. The column carries an axial load of 150,000 lb. By means of the formula $P/A = 16{,}000 - 70\ l/k$, compute the load W, eccentric 10⅝ in., which can be applied in addition. *Ans.* 96,650 lb.

3. Compute the total load the column described in Problem 2 will carry if it has no eccentricity. Assuming that the 150,000 lb. remains axial, compute the eccentricity along the web that the remaining part of the load may have safely. *Ans.* 343,000 lb.; 2.6 in.

4. Compute the diameter of a circular steel column 15 ft. long to carry a load of 72,000 lb. applied with an eccentricity of 3 in., by means of the formula of the American Institute of Steel Construction. *Ans.* 6.41 in.

CHAPTER XVII

DEFLECTION OF BEAMS BY AREA MOMENT METHOD

129. Derivation of Area Moment Equations.—The method of obtaining the deflections of a beam by means of the moment of the area of the moment diagram has decided advantages in certain cases. The fundamental equations for this method will, therefore, be derived here and their application to a few cases will be shown.

It was shown in Art. 58 that the moment M at any point in a beam is equal to the product of EI and the second derivative of y with respect to x, M being positive if the beam is concave upward, x being positive to the right, and y positive upward.

$$EI\frac{d^2y}{dx^2} = M.$$

$$EId\left(\frac{dy}{dx}\right) = Mdx.$$

If the moment of inertia I is a constant,

$$EI\int d\left(\frac{dy}{dx}\right) = \int Mdx.$$

The quantity $\int d\left(\frac{dy}{dx}\right)$, if integrated between the limits dy/dx at A, Fig. 176, and dy/dx at B, becomes the change in slope of the elastic curve between A and B.

If θ is the angle between the tangent to the elastic curve at A and that at B, the expression $EI\int d\left(\frac{dy}{dx}\right)$ becomes $EI\theta$, the angle θ being so small that tan θ may be taken as equal to θ with no appreciable error.

It will be seen in the moment diagram, Fig. 176, that the quantity Mdx is the differential area of the moment diagram. If the quantity Mdx is integrated between the limits x_A and x_B, it will, therefore, give the area $ABCD$.

$$EI\theta = \text{area } ABCD.$$

The area of the moment diagram between any two points is equal to EI times the change in slope of the elastic curve between those two points.

If the slope at the fixed end of a cantilever beam is zero, the slope at the free end is equal to $1/EI$ times the entire area of the moment diagram.

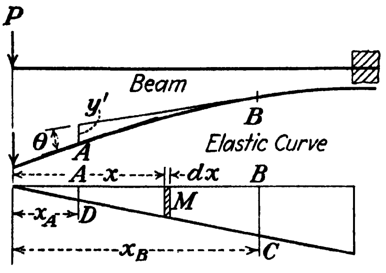


FIG. 176.

In case the moment of inertia I is not constant, the expression becomes,

$$E\int d\left(\frac{dy}{dx}\right) = \int \frac{M}{I}dx.$$

In this case, the M/I diagram must be plotted instead of the moment diagram. Then the area of the M/I diagram between any two points is equal to E times the change in slope of the elastic curve between those two points.

Referring again to the case in which the moment of inertia I is constant,

$$EId\left(\frac{dy}{dx}\right) = Mdx.$$

Since dy/dx is $\tan \theta$ and since, for small angles, $\tan \theta$ may be taken as equal to θ,

$$EId\theta = Mdx.$$

The deflection y' of the point A from the tangent at B is

$$y' = \int dy = \int (x - x_A)d\theta..$$

$$EIy' = \int_{x_A}^{x_B}(x - x_A)EId\theta.$$

$$EIy' = \int_{x_A}^{x_B}(x - x_A)Mdx:$$

The quantity Mdx is the area of the differential strip of the moment diagram. The quantity $(x - x_A)Mdx$ is the moment of this differential area with respect to the point A. The quantity

$$\int_{x_A}^{x_B}(x - x_A)Mdx$$

is, therefore, the moment of the area of the moment diagram between A and B with respect to point A.

The moment of the area of the moment diagram between any two points A and B with respect to point A is equal to EI times the deflection of point A from the tangent at point B.

If point A is at the free end of a cantilever beam and point B is at the fixed end, the deflection y of the free end from the tangent at the fixed end is equal to $1/EI$ times the moment of the area of the moment diagram with respect to the free end.

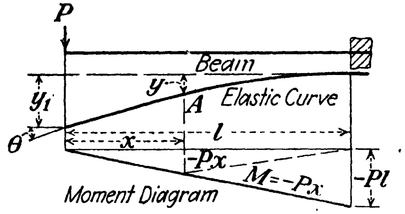


FIG. 177.

130. Cantilever Beam with Concentrated Load at the Free End.—Figure 177 represents a cantilever beam of length l with a load P at the free end. Below the beam the moment diagram is drawn to scale. The moment diagram is a triangle and its area is $-Pl^2/2$. By the first principle derived in Art. 129, the area of the moment diagram times $1/EI$ when added to the slope θ of the elastic curve at the free end will give the slope at the fixed end. Since the slope at the fixed end is 0, and the area of the moment diagram is negative,

$$\theta - \frac{Pl^2}{2EI} = 0.$$

$$\theta = \frac{Pl^2}{2EI}.$$

At any point A on the beam, the slope θ_A with respect to the fixed end is equal to $1/EI$ times the area of the trapezoid between A and the fixed end.

$$\theta_A = \frac{P}{2EI}(l^2 - x^2).$$

The change in slope between the free end and point A is given by the quantity $1/EI$ times the area of the triangle between A and the end.

$$\theta - \theta_A = \frac{Px^2}{2EI}.$$

By the second principle derived in Art. 129, the deflection y_1 of the free end of the cantilever from the tangent at the fixed end, is given by the quantity $1/EI$ times the moment of the area of the moment diagram with respect to the free end. The distance of the centroid of the triangle from the vertex is $\frac{2}{3}l$ and the area of the triangle is $-Pl^2/2$. Therefore,

$$y_1 = -\frac{Pl^2}{2} \times \frac{2}{3}l \times \frac{1}{EI}.$$

$$y_1 = -\frac{Pl^3}{3EI}.$$

The deflection y of any point A is given by the quantity $1/EI$ times the moment of the area of the trapezoid with respect to A. The dotted line divides the trapezoid into two triangles. Then

$$y = -\frac{Px}{2EI} \times \frac{(l-x)^2}{3} - \frac{Pl}{2EI} \times \frac{2}{3}(l-x)^2.$$

$$y = -\frac{P}{6EI}(x^3 - 3l^2x + 2l^3).$$

The equation just derived is the equation of the elastic curve of the beam.

PROBLEMS

1. Compute the slope and the deflection at the middle of a cantilever beam of length l with a load P at the free end.

Ans. $\theta = \frac{3}{8}\frac{Pl^2}{EI}$; $y = \frac{5}{48}\frac{Pl^3}{EI}$.

2. A wooden cantilever beam is 6 in. wide, 16 in. deep, and 10 ft. long. Compute the load at the free end which will cause a deflection of ¾ in. if the value of E is 1,500,000 lb./in.². *Ans.* 4000 lb.

3. If the allowable stress in the beam described in Problem 2 is 1400 lb./in.², what will be the amount of the slope and the deflection at the end when the beam is loaded to its allowable value? *Ans.* 0.007; 0.56 in.

4. Locate the point on a beam of length l for which the deflection is one-half of the amount that it is at the end. *Ans.* 0.347l.

5. A 5-in. 10-lb. I-beam 10 ft. long is used as a cantilever. If $E = 29{,}000{,}000$ lb./in.^{2}, what load at the free end will cause a deflection of 0.3 in. at a point 4 ft. from the free end? *Ans.* 423 lb.

131. Cantilever Beam with Concentrated Load at Any Point.—

Figure 178 represents a cantilever beam of length l with a load P at a distance of a from the free end and b from the fixed end.

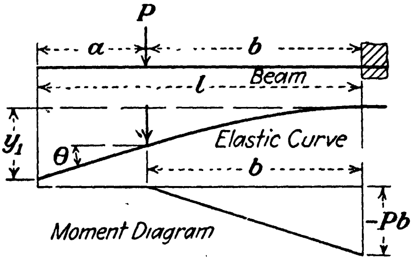


FIG. 178.

The area of the moment diagram is $-Pb^2/2$, so the slope of the beam under the load and from there to the free end is,

$$\theta = \frac{Pb^2}{2EI}.$$

The distance of the centroid of the triangle from the free end of the beam is $a + \frac{2}{3}b$, so the deflection at the end is,

$$y_1 = -\frac{Pb^2}{2EI}\left(a + \frac{2}{3}b\right).$$

If there are several concentrated loads on a beam, the slope and the deflection at any point due to each load may be obtained separately. The total slope is the sum of all of the separate slopes and the total deflection is the sum of all of the separate deflections.

PROBLEMS

1. A steel shaft 2 in. in diameter and 30 in. long is fixed at one end and carries a load of 400 lb. at a point 10 in. from the free end. Compute the slope and the deflection at the free end if $E = 30{,}000{,}000$ lb./in.^{2}.
Ans. 0.00339; 0.0792 in.

2. If the load on the beam described in Problem 1 is a movable load, how far from the free end must it be placed to cause a deflection of 0.1 in. at the free end? *Ans.* 7 in.

3. A wooden cantilever beam 8 in. wide, 2 in. deep, and 6 ft. long has a load of 100 lb. at the middle. What additional load may be applied at the free end if the limiting deflection at the free end is 1 in. and $E = 1{,}500{,}000$ lb./in.^{2}? *Ans.* 33 lb.

132. Cantilever Beam with Uniformly Distributed Load.—Figure 179 represents a cantilever beam of length l with a uniformly distributed load of w per linear unit. The moment curve is a parabola and the area enclosed by the axis and the curve is equal to one-third of the circumscribing parallelogram, $-wl^2/2 \times l/3$, or $-wl^3/6$.

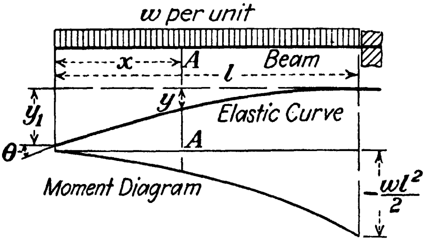


FIG. 179.

Since the slope of the beam at the fixed end is zero, and since the quantity $-wl^3/6EI$ added to θ must equal zero,

$$\theta - \frac{wl^3}{6EI} = 0.$$

$$\theta = \frac{wl^3}{6EI}.$$

The centroid of the area of the moment diagram is located at a point $\frac{3}{4}l$ from the vertex, so the moment of the area with respect to the vertex is $-wl^3/6 \times 3l/4$.

$$y_1 = -\frac{wl^4}{8EI}.$$

The deflection y at A is given by the quantity $1/EI$ times the moment with respect to point A, of the area of the moment diagram to the right of A. The moment of this area is obtained by subtracting the moment of the area to the left of A from the moment of the total area. The distance of the centroid of the total area from point A is $\frac{3}{4}l - x$, and that of the area on the left of A is $-x/4$.

$$y = \frac{1}{EI}\left(-\frac{wl^3}{6}\right)\left(\frac{3}{4}l - x\right) - \frac{1}{EI}\left(-\frac{wx^3}{6}\right)\left(-\frac{x}{4}\right).$$

$$y = \frac{w}{EI}\left(-\frac{l^4}{8} + \frac{l^3x}{6} - \frac{x^4}{24}\right).$$

The expression just derived is the equation of the elastic curve of the beam.

PROBLEMS

1. Steel reinforcing bars 1 in. square and 30 ft. long are balanced at the middle. Compute the distance the ends deflect below the level of the middle, if $E = 29{,}000{,}000$ lb./in.². *Ans.* 15.4 in.

2. A wooden cantilever beam 4 in. wide, 12 in. deep, and 8 ft. long carries a uniformly distributed load sufficient to cause a maximum flexural stress of 1250 lb./in.². Using $E = 1{,}500{,}000$ lb./in.², compute the deflection at the end. *Ans.* 0.32 in.

3. An 8-in. 18.4-lb. I-beam 10 ft. long used as a cantilever beam carries a total uniformly distributed load of 4620 lb. including its own weight. Using E as 29,000,000 lb./in.², compute the slope and the deflection at the free end. *Ans.* 0.00673; 0.605 in.

133. Cantilever Beam with Uniformly Distributed Load over Part of Beam.—Figure 180 represents a cantilever beam of length

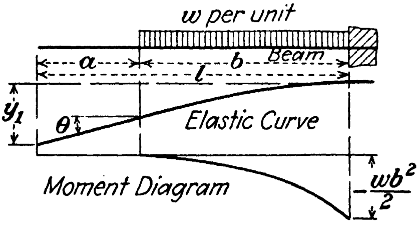


FIG. 180.

l loaded with a uniformly distributed load of w per linear unit over a length of b from the fixed end, the remaining length a having no load. The area of the moment diagram is $-wb^3/6$, so the slope at the end of the load and from there to the end of the beam is,

$$\theta = \frac{wb^3}{6EI}.$$

The distance from the free end of the beam to the centroid of the area of the moment diagram is $a + \frac{3}{4}b$. The deflection of the free end of the beam below the level of the fixed end is, therefore,

$$y_1 = -\frac{wb^3}{6EI}\left(a + \frac{3}{4}b\right).$$

Figure 181 represents a cantilever beam of length l loaded with a uniformly distributed load of w per unit over a length of a from the free end, the remaining length of b having no load. The area of the moment diagram for the length a is $-wa^3/6$, and that for the remaining length b is $-wabl/2$. The slope of the beam at the right end of the load is, therefore,

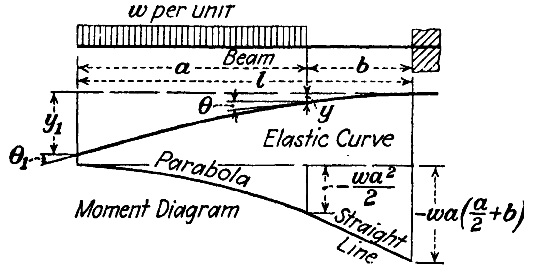


FIG. 181.

$$\theta = \frac{wabl}{2EI},$$

and that at the free end of the beam is,

$$\theta_1 = \frac{wabl}{2EI} + \frac{wa^3}{6EI}.$$

The deflection of the beam at the right end of the load is,

$$y = -\frac{wab}{12EI}(3ab + 4b^2).$$

The deflection at the end of the beam is,

$$y_1 = -\frac{wa}{24EI}(3a^3 + 12a^2b + 18ab^2 + 8b^3).$$

PROBLEMS

1. Compute the slope and the deflection at the end of the beam shown in Fig. 180 if a and b are equal. *Ans.* $\theta = \frac{wl^3}{48EI}$; $y_1 = -\frac{7}{384}\frac{wl^4}{EI}$.

2. Compute the slope and the deflection at the end of the beam shown in Fig. 181 if a and b are equal. *Ans.* $\theta_1 = \frac{7}{48}\frac{wl^3}{EI}$; $y_1 = -\frac{41}{384}\frac{wl^4}{EI}$.

3. A wooden cantilever beam 4 in. wide, 10 in. deep, and 12 ft. long carries a load of 200 lb. per linear ft. over a length of 8 ft. from the free end. Using E as 1,500,000 lb./in.², compute the deflection of the free end due to the superimposed load. *Ans.* −1.71 in.

4. Solve Problem 3 if the load extends 8 ft. from the fixed end. *Ans.* −0.590 in.

5. A steel rod 1 in. in diameter and 20 in. long is fixed at one end and carries a total load of 240 lb. distributed uniformly over a length of 8 in. at the middle of the beam. Using the value of E as 30,000,000 lb./in.² compute the deflection of the free end due to the 240-lb. load. *Ans.* 0.140 in.

134. Simple Beam with Concentrated Load at the Middle.—

Figure 182 represents a simple beam of length l supported at the ends and loaded with a concentrated load P at the middle. Each reaction is $P/2$ and the moment increases uniformly from zero at each end to a maximum value of $+Pl/4$ at the middle. The slope of the beam under the load P is zero. By the first principle derived in Art. 129, the slope θ at the left end of the beam plus $1/EI$ times the area of the moment diagram between the left end and the middle is equal to the slope at the middle.

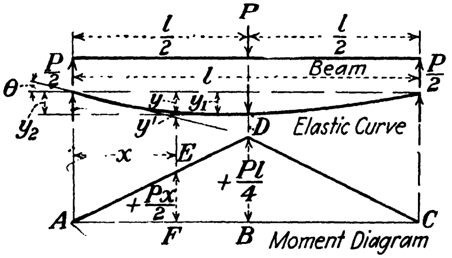


FIG. 182.

$$\theta + \frac{Pl}{4EI} \times \frac{l}{4} = 0.$$

$$\theta = -\frac{Pl^2}{16EI}.$$

By the second principle derived in Art. 129, the deflection upward of the left end from the tangent at the middle is equal to $1/EI$ times the moment of the area ABD with respect to point A.

$$y_2 = \frac{Pl^2}{16EI} \times \frac{l}{3} = \frac{Pl^3}{48EI}.$$

The deflection y_1 of the middle below the level of the ends is the same as y_2 in amount, but is, of course, negative.

$$y_1 = -\frac{Pl^3}{48EI}.$$

At point E, the distance of the tangent at the left end from the horizontal line through the left end is,

$$x\theta = -\frac{Pl^2x}{16EI}.$$

The deflection y' of the elastic curve from the tangent at the left end is equal to $1/EI$ times the moment of the area AFE with respect to point E.

$$y' = \frac{Px^2}{4EI} \times \frac{x}{3} = \frac{Px^3}{12EI}.$$

The deflection y of the elastic curve at E below the level of the supports is, therefore,

$$y = -\frac{Pl^2x}{16EI} + \frac{Px^3}{12EI}.$$

This expression is the equation of the elastic curve of the left half of the beam.

PROBLEMS

1. Write out the expression for the slope θ at the third point.

Ans. $\theta = -\frac{5}{144}\frac{Pl^2}{EI}.$

2. Write out the expression for the deflection at the quarter point.

Ans. $y = -\frac{11}{768}\frac{Pl^3}{EI}.$

3. A wooden beam 12 in. wide, 2 in. deep, and 16 ft. long is supported at the ends. If E is 1,600,000 lb./in.^{2}, what load at the middle will cause a deflection of 2 in.? At what distance from the end is the deflection 1 in.?

Ans. 173.6 lb.; 33.3 in.

135. Simple Beam with Concentrated Load at Any Point.—Figure 183 represents a simple beam of length l with a load P at a distance of a from one end and a distance of b from the other end, a being greater than b. By the second principle of Art. 129 the deflection upward of the elastic curve at the right end from the tangent at the left end is equal to $1/EI$ times the moment of the area $ABCD$ of the moment diagram with respect to point C. The area of the triangle ABD is $Pa^2b/2l$ and its centroid is distant $a/3 + b$ from point C. The area of the triangle CBD is $Pab^2/2l$ and its centroid is distant $\frac{2}{3}b$ from point C. The deflection is, therefore,

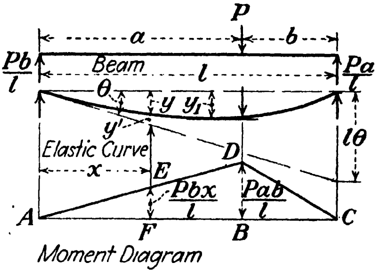


FIG. 183.

$$l\theta = \frac{Pab}{EIl}\left(\frac{a^2}{6} + \frac{ab}{2} + \frac{b^2}{3}\right).$$

The slope at the left end is,

$$\theta = -\frac{Pab}{6EIl^2}(a^2 + 3ab + 2b^2).$$

This equation may be reduced to a simpler form by factoring, since $a^2 + 3ab + 2b^2 = (a + b)(a + 2b)$ and $a + b = l$.

$$\theta = -\frac{Pab}{6EIl}(a + 2b).$$

The slope at any other point on the beam is equal to the slope at the left end plus $1/EI$ times the area of the moment diagram up to that point. At point E the slope is,

$$\theta_E = -\frac{Pab}{6EIl}(a + 2b) + \frac{Pbx^2}{2EIl}.$$

At the point of maximum deflection the slope is zero, so,

$$-\frac{Pab}{6EIl}(a + 2b) + \frac{Pbx^2}{2EIl} = 0.$$

$$x = \sqrt{\frac{a^2 + 2ab}{3}}.$$

At point E, distant x from the left end, if x is less than a, the deflection y' of the elastic curve from the tangent at the end is given by the quantity $1/EI$ times the moment of the area AFE with respect to point E.

$$y' = \frac{Pbx^2}{2EIl} \times \frac{x}{3} = \frac{Pbx^3}{6EIl}.$$

The deflection y of the elastic curve from the horizontal line through the ends is given by the expression,

$$y = x\theta + y'.$$

$$y = -\frac{Pabx}{6EIl}(a + 2b) + \frac{Pbx^3}{6EIl}.$$

$$y = -\frac{Pbx}{6EIl}(a^2 + 2ab - x^2).$$

At the point under the load where $x = a$, the expression for y reduces to,

$$y_P = -\frac{Pa^2b^2}{3EIl}.$$

The maximum deflection y_1 is at the point where $x = \sqrt{\frac{a^2 + 2ab}{3}}$.

$$y_1 = -\frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27EIl}.$$

The point of maximum deflection is at the greatest distance from the middle when b approaches zero in value, and a approaches l. For these values,

$$x = \frac{l}{\sqrt{3}} = 0.577l.$$

It is seen from this that the point of maximum deflection is never farther than $0.077l$ from the middle of the beam. The deflection at the middle of the beam may be used instead of the maximum unless extraordinary accuracy is required.

PROBLEMS

1. Derive the expression for the deflection at the middle of the beam.

Ans. $y_0 = -\frac{Pb}{48EI}(3l^2 - 4b^2)$.

2. Derive the expression for the slope at the right end of the beam.

Ans. $\frac{Pab}{6EIl}(2a + b)$.

3. Derive the expression for the deflection of the beam between the load and the right end in terms of x measured from the right end, positive to the left.

Ans. $y = -\frac{Pax}{6EIl}(b^2 + 2ab - x^2)$.

4. A steel shaft 3 in. in diameter and 20 ft. long, supported at the ends, carries a load of 150 lb. at a point 6 ft. from one end. Using $E = 30{,}000{,}000$

lb./in.², compute the maximum deflection, the deflection at the middle, and the deflection at the load due to the load alone.

Ans. 0.290 in.; 0.287 in.; 0.255 in.

5. A 24-in. 100-lb. I-beam 50 ft. long is supported at the ends and carries a load of 24,000 lb. at a point 15 ft. from one end. Using $E = 29{,}000{,}000$ lb./in.², compute the maximum deflection and the deflection at the middle due to the concentrated load. *Ans.* 1.26 in.; 1.24 in.

136. Simple Beam with Uniformly Distributed Load.

—Figure 184 represents a simple beam with a uniformly distributed load of w per linear unit. The slope of the tangent at the left end of the beam is obtained from the area of the moment diagram. Since by symmetry the slope at the middle is zero,

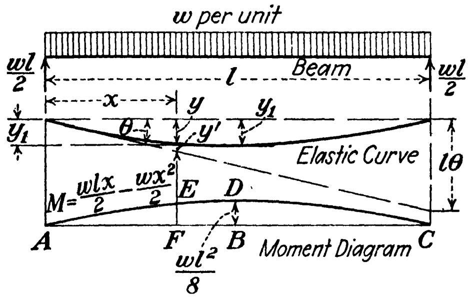


FIG. 184.

$$\theta + \frac{1}{EI} \times \text{area } ABD = 0.$$

$$\theta + \frac{wl^3}{24EI} = 0.$$

$$\theta = -\frac{wl^3}{24EI}.$$

The maximum deflection, y_1, of the middle below the ends is the same as the deflection of the end upward from the tangent at the middle, and this is given by the quantity $1/EI$ times the moment of the area ABD with respect to the point A. The distance of the centroid of this area from the center line DB is three-eighths of length AB, so the distance from point A is $\frac{5}{8}AB$ or $\frac{5}{16}l$.

$$y_1 = \frac{wl^3}{24EI} \times \frac{5}{16}l = \frac{5}{384}\frac{wl^4}{EI}.$$

The deflection of the middle below the ends is, of course, negative.

The deflection upward of the elastic curve from the tangent at the left end is given by the quantity $1/EI$ times the moment of the area AFE with respect to point F. In Fig. 185 this area AFE is shown separated into its two component areas, the positive triangle GHI and the negative parabolic area GHJ. The amount of the area GHI is $wlx^2/4$, and the distance from IH to its centroid is $x/3$. The amount of the area GHJ is $wx^3/6$ and the distance from IH to its centroid is $x/4$.

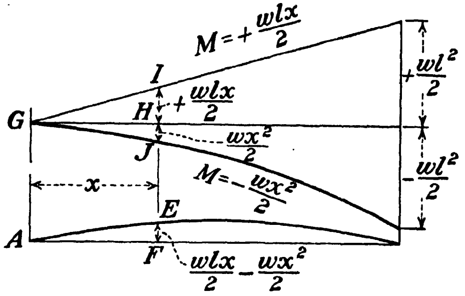


FIG. 185.

$$y' = \frac{wlx^2}{4EI} \times \frac{x}{3} - \frac{wx^3}{6EI} \times \frac{x}{4}.$$

$$y' = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI}.$$

The deflection y of the elastic curve below the level of the ends is, therefore, the algebraic sum of θx and y'.

$$y = \theta x + y'.$$

$$y = -\frac{wl^3x}{24EI} + \frac{wlx^3}{12EI} - \frac{wx^4}{24EI}.$$

$$y = -\frac{wx}{24EI}(l^3 - 2lx^2 + x^3).$$

PROBLEMS

1. Compute the slope and the deflection of the simple beam with uniformly distributed load at a point $l/4$ from the right end.

Ans. $\frac{11}{384}\frac{Wl^2}{EI}; -\frac{19}{2048}\frac{Wl^3}{EI}.$

2. A 24-in. 79.9-lb. I-beam, 40 ft. long, is supported at the ends and carries a total uniformly distributed load of 50,000 lb. Using E as 29,000,000 lb./in.^{2} compute the maximum deflection. *Ans.* 1.19 in.

137. Simple Beam with Two Symmetrical Loads.—Figure 186 represents a simple beam of length l, loaded with two concentrated loads. Each load is P in amount and is placed at a distance a from the end. The distance between the loads is b.

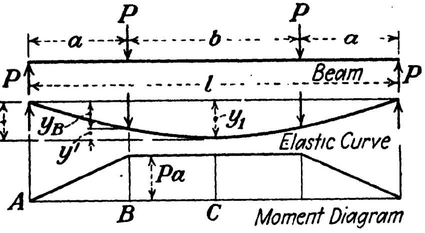


FIG. 186.

By symmetry, the slope of the beam at the middle is zero. The slope θ_B at B plus $1/EI$ times the area between B and C is equal to zero.

$$\theta_B + \frac{Pab}{2EI} = 0.$$

$$\theta_B = -\frac{Pab}{2EI}.$$

In a similar manner, the slope θ_A at the left end is obtained.

$$\theta_A = -\frac{Pab}{2EI} - \frac{Pa^2}{2EI}.$$

$$\theta_A = -\frac{Pa}{2EI}(a + b).$$

The deflection upward of the left end from the tangent at the middle is given by the quantity $1/EI$ times the moment of the area between A and C with respect to point A.

$$y_A = \frac{Pa^2}{2EI} \times \frac{2a}{3} + \frac{Pab}{2EI}\left(a + \frac{b}{4}\right).$$

The deflection of the middle point below the level of the ends is the same in amount, but negative in sign.

$$y_1 = -\frac{Pa}{EI}\left(\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{8}\right).$$

The deflection upward of the point of application of the load P above the tangent at the middle is given by the quantity $1/EI$

times the moment of the area between B and C with respect to point B.

$$y' = \frac{Pab}{2EI} \times \frac{b}{4} = \frac{Pab^2}{8EI}.$$

The deflection y_B of the point of application of the load P below the level of the ends is given by the algebraic sum of y_1 and y'.

$$y_B = -\frac{Pa}{EI}\left(\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{8}\right) + \frac{Pab^2}{8EI}.$$

$$y_B = -\frac{Pa^2}{EI}\left(\frac{a}{3} + \frac{b}{2}\right).$$

PROBLEMS

1. Derive the general expression for the slope θ between A and B.

Ans. $\theta = -\frac{P}{2EI}(a^2 + ab - x^2)$.

2. A wooden beam 12 in. wide, 18 in. deep, and 24 ft. long, supported at the ends, carries two symmetrically placed loads 16 ft. apart of 15,000 lb. each. Compute the slope under the load, the deflection under the load and the maximum deflection. Use $E = 1{,}500{,}000$ lb./in.^{2}.

Ans. 0.0079; 0.443 in.; 0.821 in.

CHAPTER XVIII

CURVED BEAMS AND HOOKS

138. Location of Neutral Axis.—If a beam is curved so that the fibers which are in tension differ appreciably in length from those in compression, the distribution of stress is different from what it was assumed to be in Chap. VI. For curved beams the same as for straight beams, the assumption is made that the deformation of any fiber is proportional to its distance from the neutral axis. This means that any cross section of the beam remains plane after loading, but the neutral axis of any cross section is not the centroidal axis.

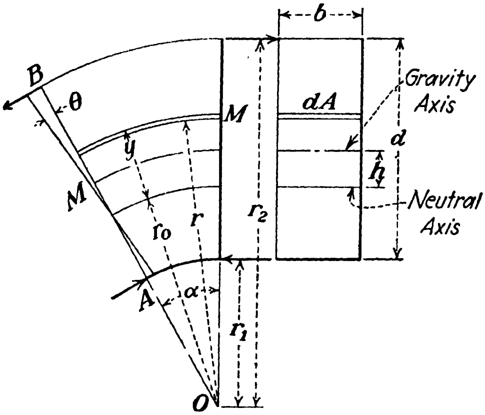


FIG. 187.

Figure 187 represents a portion of a curved beam between two planes, both of which pass through the center of curvature O. The inner radius is r_1 and the outer radius is r_2. The beam is subjected to the action of equal and opposite couples, each of magnitude M. Since there is zero resultant force across any cross section such as AB, and since the unit deformation and, therefore, the unit stress is greater on the concave side, the position of the neutral axis is shifted a distance h from the centroidal axis toward the concave side of the beam.

Let θ be the angle through which the plane AB is turned by the moment M. If y is the distance of any fiber from the neutral axis, the total deformation of the fiber is $y\theta$ and the unit deformation is

$$\delta = \frac{y\theta}{r\alpha}.$$

The unit stress is,

$$S = E\delta = \frac{E\theta y}{r\alpha}.$$

The force acting on area dA is,

$$dF = \frac{E\theta y dA}{r\alpha}.$$

Since the total force across the plane AB is zero,

$$F = \frac{E\theta}{\alpha}\int\frac{y}{r}dA = 0.$$

Since $y = r - r_0$,

$$\int_{r_1}^{r_2}\frac{(r - r_0)}{r}dA = 0.$$

$$\int_{r_1}^{r_2}dA - r_0\int_{r_1}^{r_2}\frac{dA}{r} = 0.$$

$$A = r_0\int_{r_1}^{r_2}\frac{dA}{r}.$$

$$r_0 = \frac{A}{\int_{r_1}^{r_2}\frac{dA}{r}}.$$

139. General Expression for Relation between Stress and Moment.—The cross section of the beam represented in Fig. 187 is rectangular, but the following derivation is valid for a cross section of any shape.

From Art. 138, the differential force acting on the area dA, Fig. 187, is

$$dF = \frac{E\theta y dA}{r\alpha}.$$

The moment of this differential force with respect to an axis through point O is

$$dM = \frac{E\theta}{\alpha} y dA.$$

The total moment of all of these differential forces constitutes the couple M.

$$M = \frac{E\theta}{\alpha}\int y dA.$$

$$M = \frac{E\theta}{\alpha}\bar{y}A.$$

But $\bar{y} = h$, so

$$M = \frac{E\theta}{\alpha}Ah.$$

$$\frac{E\theta}{\alpha} = \frac{M}{Ah}.$$

From Art. 138,

$$S = \frac{E\theta}{\alpha}\frac{y}{r}.$$

$$S = \frac{My}{Ahr}.$$

Though the total deformation of any fiber varies directly as its distance from the neutral axis, the *unit* deformation and therefore the unit stress do not vary directly as the distance from the neutral axis on account of the difference in the original length of the fibers.

140. Stresses in Curved Beams of Rectangular Cross Section.—If the cross section of a curved beam is rectangular, the width being b and the depth $d = r_2 - r_1$, $dA = b\,dr$, and the expression for r_0 as derived in Art. 138 becomes

$$r_0 = \frac{bd}{b\int_{r_1}^{r_2}\frac{dr}{r}}.$$

$$r_0 = \frac{d}{\log_e \frac{r_2}{r_1}}.$$

From Fig. 187,

$$h = r_1 + \frac{d}{2} - r_0.$$

EXAMPLE

A curved beam is 4 in. wide, 12 in. deep, and has a radius of 3 in. at the inner surface. Compute r_0 and h. Compute the unit stresses at the inner fiber, and at 1, 2, 3, 4, 6, 8, 10, and 12 in. from the inner fiber across the cross section, due to a moment of 60,000 ft.-lb. acting to cause compression at the inner fiber.

Solution:

$$\begin{aligned} r_0 &= \frac{12}{\log_e 5} = \frac{12}{1.609} = 7.458 \text{ in.} \\ h &= 9 - 7.458 = 1.542 \text{ in.} \\ M &= 60{,}000 \times 12 = 720{,}000 \text{ in.-lb.} \\ A &= 48 \text{ sq. in.} \end{aligned}$$

At the inner fiber,

$$S = \frac{720{,}000 \times 4.458}{48 \times 1.542 \times 3} = 14{,}460 \text{ lb./in.}^2 \text{ C.}$$

$$\text{At 1 in., } S = \frac{720{,}000 \times 3.458}{48 \times 1.542 \times 4} = 8410 \text{ lb./in.}^2 \text{ C.}$$

$$\text{At 2 in., } S = \frac{720{,}000 \times 2.458}{48 \times 1.542 \times 5} = 4780 \text{ lb./in.}^2 \text{ C.}$$

$$\text{At 3 in., } S = \frac{720{,}000 \times 1.458}{48 \times 1.542 \times 6} = 2360 \text{ lb./in.}^2 \text{ C.}$$

$$\text{At 4 in., } S = \frac{720{,}000 \times 0.458}{48 \times 1.542 \times 7} = 640 \text{ lb./in.}^2 \text{ C.}$$

$$\text{At 6 in., } S = \frac{720{,}000 \times 1.542}{48 \times 1.542 \times 9} = 1670 \text{ lb./in.}^2 \text{ T.}$$

$$\text{At 8 in., } S = \frac{720{,}000 \times 3.542}{48 \times 1.542 \times 11} = 3140 \text{ lb./in.}^2 \text{ T.}$$

$$\text{At 10 in., } S = \frac{720{,}000 \times 5.542}{48 \times 1.542 \times 13} = 4150 \text{ lb./in.}^2 \text{ T.}$$

$$\text{At 12 in., } S = \frac{720{,}000 \times 7.542}{48 \times 1.542 \times 15} = 4890 \text{ lb./in.}^2 \text{ T.}$$

The variation in the unit stress across the cross section is shown graphically in Fig. 188.

PROBLEMS

1. Solve the example above for the stresses at the inner and outer fibers if the radius of the inner surface is 1.5 in.
Ans. $r_0 = 5.46$ in.; $h = 2.04$ in.; $S_1 = 19{,}410$ lb./in.^{2}; $S_2 = 4380$ lb./in.^{2}.

2. A curved cast-iron beam is 3 in. square and has a radius of 1 in. at the inner surface. If the allowable unit stress in tension is 3000 lb./in.^{2} and that in compression is 16,000 lb./in.^{2}, compute the allowable moment if the concave surface is in compression. Compute the allowable moment if the concave surface is in tension. *Ans.* 19,960 in.-lb.; 7910 in.-lb.

141. Stresses in Curved Beams of T-section.—If the cross section of a curved beam is a T-section, two sets of limits are required in the integration of the expression $\int \frac{dA}{r}$.

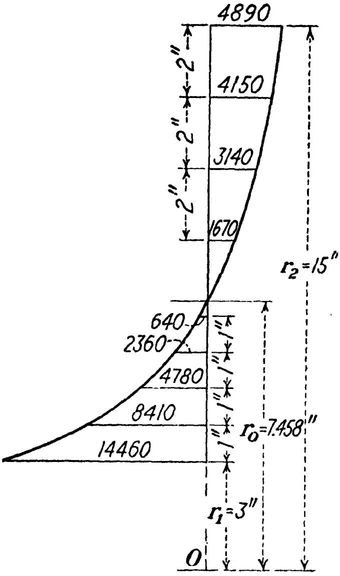


FIG. 188.

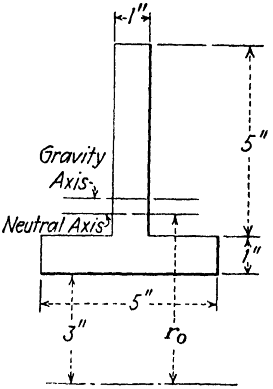


FIG. 189.

EXAMPLE

Figure 189 represents a T-beam with the flange on the concave side. Compute r_0 and h. Compute the stresses S_1 and S_2 due to a moment of 100,000 in.-lb.

Solution:

$$r_0 = \frac{A}{\int \frac{dA}{r}}.$$

$$r_0 = \frac{10}{5 \log_e \frac{4}{3} + \log_e \frac{9}{4}} = 4.446 \text{ in.}$$

$$\bar{y} = \frac{5 \times \frac{1}{2} + 5 \times 3\frac{1}{2}}{10} = 2 \text{ in.}$$

$$h = 2 - 1.446 = 0.554 \text{ in.}$$

$$S_1 = \frac{100{,}000 \times 1.446}{10 \times 0.554 \times 3} = 8700 \text{ lb./in.}^2.$$

$$S_2 = \frac{100{,}000 \times 4.554}{10 \times 0.554 \times 9} = 9130 \text{ lb./in.}^2.$$

PROBLEM

1. Solve the example above if the flange is on the convex side of the beam and the radius of curvature of the inside surface is changed to 2 in.

Ans. $r_0 = 5.2$ in.; $h = 0.8$ in.; $S_1 = 20{,}000$ lb./in.^{2}; $S_2 = 4375$ lb./in.^{2}.

142. Stresses in Curved Beams of Circular Section.—In the circular cross section shown in Fig. 190, the radius of the circle is a.

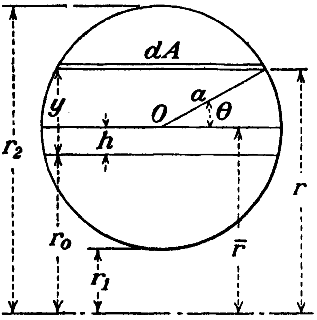


FIG. 190.

$$A = \pi a^2.$$
$$dA = 2a \cos \theta dr.$$
$$r = \bar{r} + a \sin \theta.$$
$$dr = a \cos \theta d\theta.$$
$$dA = 2a^2 \cos^2 \theta d\theta.$$

The expression $\int \frac{dA}{r}$ then becomes,

$$\int \frac{dA}{r} = 2\int \frac{a^2 \cos^2 \theta}{\bar{r} + a \sin \theta} d\theta.$$

If $\cos^2 \theta$ is replaced by $1 - \sin^2 \theta$ and the numerator is divided by the denominator, it becomes,

$$\int \frac{dA}{r} = 2\left[\bar{r}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta - a\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin \theta d\theta - (r^2 - a^2)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{d\theta}{\bar{r} + a \sin \theta}\right]$$

$$= 2\left[\pi\bar{r} + 0 - 2\sqrt{\bar{r}^2 - a^2}\left(\tan^{-1} \frac{a + \bar{r}}{\sqrt{\bar{r}^2 - a^2}} - \tan^{-1} \frac{a - \bar{r}}{\sqrt{\bar{r}^2 - a^2}}\right)\right].$$

Since $\tan^{-1} \frac{a + \bar{r}}{\sqrt{\bar{r}^2 - a^2}}$ and $\tan^{-1} \frac{a - \bar{r}}{\sqrt{\bar{r}^2 - a^2}}$ are numerically complementary angles, and since $\tan^{-1} \frac{a - \bar{r}}{\sqrt{\bar{r}^2 - a^2}}$ is negative, their difference is always equal to $\frac{\pi}{2}$.

$$\int \frac{dA}{r} = 2\pi(\bar{r} - \sqrt{\bar{r}^2 - a^2}).$$

$$r_0 = \frac{A}{\int \frac{dA}{r}} = \frac{a^2}{2(\bar{r} - \sqrt{\bar{r}^2 - a^2})}.$$

$$r_0 = \tfrac{1}{2}(\bar{r} + \sqrt{\bar{r}^2 - a^2}).$$

EXAMPLE

A curved beam of circular cross section is 6 in. in diameter and has an inner radius of 8 in. Compute r_0 and h. Compute the stresses S_1 and S_2 due to a moment of 200,000 in.-lb.

Solution:

$$\bar{r} = 11 \text{ in.}$$

$$r_0 = \tfrac{1}{2}(11 + \sqrt{121 - 9}) = 10.7915 \text{ in.}$$

$$h = 0.2085 \text{ in.}$$

$$S_1 = \frac{200{,}000 \times 2.7915}{28.27 \times 0.2085 \times 8} = 11{,}830 \text{ lb./in.}^2.$$

$$S_2 = \frac{200{,}000 \times 3.2085}{28.27 \times 0.2085 \times 14} = 7770 \text{ lb./in.}^2.$$

PROBLEMS

1. A curved beam of circular cross section is 4 in. in diameter and has an inner radius of 1 in. Compute r_0 and h. Compute S_1 and S_2 due to a moment of 60,000 in.-lb.

Ans. $r_0 = 2.618$ in.; $h = 0.382$ in.; $S_1 = 20{,}220$ lb./in.^{2}; $S_2 = 5960$ lb./in.^{2}.

2. A curved beam of circular cross section is 8 in. in diameter and has an inner radius of 2 in. If the allowable unit stress is 15,000 lb./in.^{2}, what is the maximum allowable moment? *Ans.* 29,670 ft.-lb.

143. Stresses in Curved Bars and Hooks of Rectangular Section.—In the discussion in the last four articles, the beams were considered to be subjected to moment only. If the load across any cross section of a beam or bar is not a couple but is a resultant force, this force may be resolved into a force through the center of gravity of the section and a couple. The moment of the couple is *Pe*, *P* being the amount of the force and *e* its eccentricity. The unit stress at any point is equal to the algebraic sum of the direct unit stress *P/A* and the unit stress due to the moment *Pe*. It is assumed that the loading is in the plane of the curvature of the bar or hook.

EXAMPLE

Figure 191 represents a curved bar of rectangular cross section with an eccentric load. Compute the unit stress at M and at N.

Solution.—The direct unit stress $P/A = 1200/6 = 200$ lb./in.². This is compression at all points in the cross section MN.

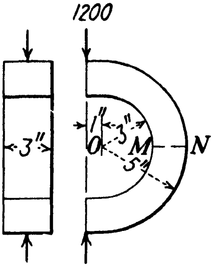


FIG. 191.

$$r_0 = \frac{2}{\log_e 5/3} = 3.9154 \text{ in.}$$
$$h = 0.0846 \text{ in.}$$
$$M = 5 \times 1200 = 6000 \text{ in.-lb.}$$

At point M, due to the moment,

$$S_1 = \frac{6000 \times 0.9154}{6 \times 0.0846 \times 3} = 3607 \text{ lb./in.}^2 \text{ C.}$$

At point N, due to the moment,

$$S_2 = \frac{6000 \times 1.0846}{6 \times 0.0846 \times 5} = 2564 \text{ lb./in.}^2 \text{ T.}$$

At point M, the total stress is

$$S_M = 3607 + 200 = 3807 \text{ lb./in.}^2 \text{ C.}$$

At point N, the total stress is

$$S_N = 2564 - 200 = 2364 \text{ lb./in.}^2 \text{ T.}$$

PROBLEMS

1. Solve the example above if the line of action of the load passes through the center O. *Ans.* $S_M = 3086$ lb./in.²; $S_N = 1851$ lb./in.².

2. A hook with a rectangular cross section is 1.6 in. wide, 5 in. deep through the shank, and has an inner radius of curvature of 3 in. Compute the stresses at the inner and outer surfaces due to a load of 8000 lb. applied on a line 2 in. from the inner face of the shank.

Ans. 8830 lb./in.² T.; 3060 lb./in.² C.

144. Stresses in Curved Bars and Hooks of Circular Section.—Loads applied to hooks are always tensile loads. Loads applied to curved bars may be either tensile or compressive loads. As in the case of curved bars of rectangular cross section, the stress at any point is the algebraic sum of the direct unit stress and the flexural unit stress.

EXAMPLE

A curved bar of circular cross section is 1 in. in diameter, and has an inner radius of curvature of 4 in. If the allowable unit stress is 10,000 lb./in.^{2} and the compressive load applied is 500 lb., what eccentricity may it have?

Solution.—The maximum unit stress is at the inner fiber.

$$a = 0.5 \text{ in.}; \qquad \bar{r} = 4.5 \text{ in.}$$
$$r_0 = \tfrac{1}{2}(4.5 + \sqrt{20.25 - 0.25}) = 4.486 \text{ in.}$$
$$h = 4.5 - 4.486 = 0.014 \text{ in.}$$

The direct unit tensile stress is

$$S = \frac{500}{0.7854} = 636 \text{ lb./in.}^2.$$
$$S_1 = 10{,}000 = \frac{500 \times 0.486 \times 4e}{\pi \times 4 \times 0.014} + 636.$$
$$e = 1.70 \text{ in.}$$

PROBLEMS

1. Compute the unit stress in the outer fiber in the bar described in the example above. *Ans.* 7310 lb./in.^{2}.

2. The shank of a hook of circular cross section is 3 in. in diameter and the inner radius of curvature is 2 in. Compute the stresses at the inner and outer surfaces due to a load of 10,000 lb. applied on a line 1.5 in. from the inner surface.
Ans. 18,150 lb./in.^{2}; 6960 lb./in.^{2}.

145. Stresses in Hooks of T-section.—

Hooks are often made with a T-section, as shown in Fig. 192. This section gives added area for wear where the load is applied, and also is a more efficient section in the shank than a plain rectangular section of the same cross-section area would be.

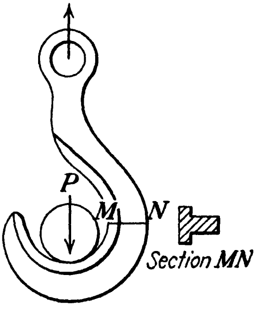


FIG. 192.

EXAMPLE

In the hook shown in Fig. 192, the flange at section MN is 2 in. wide and 1 in. deep. The web is 1.5 in. wide and 3 in. deep. The inner radius of curvature at M is 3 in., and the load P is applied along a line 2.5 in. from M. Compute the unit stresses at M and N due to a load of 10,000 lb.

Solution.—As shown in Art. 141,

$$r_0 = \frac{6.5}{2 \log_e \frac{4}{3} + 1.5 \log_e \frac{7}{4}} = 4.6 \text{ in.}$$

The distance of the centroid of the section from the inner face is

$$\bar{y} = \frac{2 \times 0.5 + 4.5 \times 2.5}{6.5} = 1.885 \text{ in.}$$
$$\bar{r} = 4.885 \text{ in.}$$
$$h = 4.885 - 4.6 = 0.285 \text{ in.}$$

The direct tensile stress is

$$S = \frac{10{,}000}{6.5} = 1540 \text{ lb./in.}^2.$$

The moment at the section is

$$M = 10{,}000(2.5 + 1.885) = 43{,}850 \text{ in.-lb.}$$
$$S_1 = \frac{43{,}850 \times 1.6}{6.5 \times 0.285 \times 3} = 12{,}620 \text{ lb./in.}^2.$$
$$S_2 = \frac{43{,}850 \times 2.4}{6.5 \times 0.285 \times 7} = 8120 \text{ lb./in.}^2.$$

The total unit stress at M is

$$S_1 + S = 12{,}620 + 1540 = 14{,}160 \text{ lb./in.}^2 \text{ T.}$$

The total unit stress at N is

$$S_2 - S = 8120 - 1540 = 6580 \text{ lb./in.}^2 \text{ C.}$$

PROBLEM

1. A hook with a T-section has a flange 4 in. wide and 1.8 in. deep, and a web 1.8 in. wide and 5 in. deep. The center of curvature of the inner surface is 3 in. If the line of action of the load is to be 4.5 in. from the inner surface of the shank and the allowable unit stress is 12,000 lb./in.², what load may be applied to the hook? *Ans.* 22,800 lb.

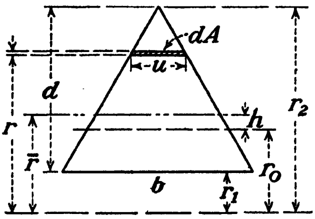


FIG. 193.

146. Stresses in Hooks of Triangular Section.—Hooks are sometimes made with a triangular cross section, as shown in Fig. 193.

$$A = \frac{bd}{2}.$$
$$dA = u\,dr.$$
$$\frac{u}{b} = \frac{r_2 - r}{d}.$$
$$u = \frac{b}{d}(r_2 - r)$$

$$\int \frac{dA}{r} = \frac{b}{d}\left(r_2 \int_{r_1}^{r_2} \frac{dr}{r} - \int_{r_1}^{r_2} dr\right),$$

$$= \frac{b}{d}\left(r_2 \log_e \frac{r_2}{r_1} - d\right).$$

$$r_0 = \frac{A}{\int \frac{dA}{r}} = \frac{d^2}{2\left(r_2 \log_e \frac{r_2}{r_1} - d\right)}.$$

EXAMPLE

In Fig. 193, let $b = 2$ in., $r_1 = 1.5$ in., $r_2 = 4$ in., and let the line of action of the load be 2 in. from the base of the cross section. Compute the maximum and minimum stresses due to a load of 4000 lb.

Solution:

$$\bar{r} = 2.333 \text{ in.}$$

$$r_0 = \frac{2.5^2}{2\left(4 \log_e \frac{4}{1.5} - 2.5\right)} = 2.19 \text{ in.}$$

$$h = 2.333 - 2.19 = 0.143 \text{ in.}$$

The direct tensile unit stress is

$$S = \frac{4000}{2.5} = 1600 \text{ lb./in.}^2.$$

The moment at the section is

$$M = 4000 \times 2.833 = 11{,}333 \text{ in.-lb.}$$

$$S_1 = \frac{11{,}333 \times 0.69}{2.5 \times 0.143 \times 1.5} = 14{,}600 \text{ lb./in.}^2.$$

$$S_2 = \frac{11{,}333 \times 1.81}{2.5 \times 0.143 \times 4} = 14{,}340 \text{ lb./in.}^2.$$

The total unit stress at the inner fiber is

$$S_1 + S = 14{,}600 + 1600 = 16{,}200 \text{ lb./in.}^2 \text{ T.}$$

The total unit stress at the outer fiber is

$$S_2 - S = 14{,}340 - 1600 = 12{,}740 \text{ lb./in.}^2 \text{ C.}$$

PROBLEM

1. The cross section of a triangular hook has a base of 2 in., an altitude of 3 in., and a radius of curvature of 2 in. at the inner face of the shank. If the allowable unit stress in tension is 18,000 lb./in.^{2}, and that in compression is 15,000 lb./in.^{2}, what load may be applied along a line 3 in. from the inner face of the shank? *Ans.* 4450 lb.

TABLES

TABLE I.—AVERAGE WEIGHTS AND SPECIFIC GRAVITIES

Material	Pounds per cubic foot	Pounds per cubic inch	Specific gravity
Timber	36	0.021	0.58
Brick	125	0.072	2.00
Brick masonry	120	0.070	1.92
Sandstone masonry	140	0.081	2.24
Limestone masonry	160	0.093	2.56
Granite masonry	165	0.096	2.64
Concrete	150	0.087	2.40
Cast iron	450	0.260	7.20
Wrought iron	480	0.278	7.68
Steel	490	0.284	7.84
Aluminum	165	0.096	2.64
Brass	535	0.310	8.55
Copper	556	0.322	8.90

TABLE II.—AVERAGE VALUES OF ELASTIC LIMIT, ULTIMATE STRENGTH, AND MODULUS OF ELASTICITY

(Pounds per Square Inch)

Material	Elastic limit		Ultimate strength		Modulus of elasticity E
	Tension	Compression	Tension	Compression	
Concrete	....	1,000	....	2,500	2,000,000
Aluminum	12,000	....	25,000	....	11,000,000
Brass	9,000	....	28,000	....	14,000,000
Copper	10,000	....	35,000	....	15,000,000
Cast iron	6,000	25,000	20,000	80,000	15,000,000
Wrought iron	27,000	27,000	50,000	50,000	27,000,000
Structural steel	35,000	35,000	65,000	65,000	29,000,000
High carbon steel (tempered)	70,000	70,000	120,000	120,000	30,000,000
Nickel steel (tempered)	120,000	120,000	150,000	150,000	30,000,000
Chrome steel (tempered)	130,000	130,000	160,000	160,000	30,000,000
Chrome vanadium steel (tempered)	170,000	170,000	240,000	240,000	30,000,000
Monarch (improved plow) wire	155,000	155,000	287,000	....	24,000,000

TABLE III.—ALLOWABLE UNIT STRESSES, NEW YORK BUILDING LAWS
(Pounds per Square Inch)

Material	Tension	Compression	Shear
Rolled structural steel	16,000	16,000	10,000
Cast steel	16,000	16,000	
Cast iron	3,000	16,000	3,000
Concrete (1:2:4)	......	500	
Brick masonry, lime mortar	......	110	
Brick masonry, cement mortar	......	250	
Sandstone	......	400	
Limestone	......	700	
Granite and slate	......	1,000	
Bluestone	......	2,000	

TABLE IV.—ALLOWABLE UNIT STRESSES IN STEEL STRUCTURES, AMERICAN BRIDGE COMPANY
(Pounds per Square Inch)

Material	Tension	Compression	Flexure	Shear	Bearing
Rolled structural steel	16,000	16,000	16,000	10,000	16,000
Cast steel	......	16,000	16,000	......	16,000
Pins	......	......	24,000	12,000	24,000
Shop rivets	......	......	......	12,000	24,000
Field rivets and bolts	......	......	......	10,000	20,000

TABLE V.—SAFE WORKING STRESSES FOR TIMBER, IN DRY LOCATIONS: FOREST PRODUCTS LABORATORY

(Pounds per Square Inch)

Species	Bending	Compression		Horizontal shear	Intermediate columns; values of K		Modulus of elasticity E
		Parallel	Normal				
	Select	Select	All	Select	Select	Common	All
Ash, black	1,000	650	300	90	26.4	29.5	1,100,000
Ash, commercial white	1,400	1,100	500	125	23.7	26.5	1,500,000
Aspen and large-tooth aspen	800	700	150	80	23.0	25.7	900,000
Basswood	800	700	150	80	23.0	25.7	900,000
Beech	1,500	1,200	500	125	23.4	26.2	1,600,000
Birch, paper	900	650	200	80	25.2	28.1	1,000,000
Birch, yellow and sweet	1,500	1,200	500	125	23.4	26.2	1,600,000
Cedar, Alaska	1,100	800	250	90	24.8	27.8	1,200,000
Cedar, western red	900	700	200	80	24.2	27.1	1,000,000
Cedar, northern and southern white	750	550	175	70	24.5	27.3	800,000
Cedar, Port Orford	1,100	900	250	90	23.4	26.2	1,200,000
Chestnut	950	800	300	90	22.7	25.3	1,000,000
Cottonwood, common and black	800	700	150	80	23.0	25.7	900,000
Cypress, bald	1,300	1,100	350	100	21.2	23.7	1,200,000
Douglas fir, coast type	1,600	1,175	345	90	23.7	27.3	1,600,000
Douglas fir, coast, dense	1,750	1,290	380	105	22.6	24.9	1,600,000
Douglas fir, Rocky Mountain type	1,100	800	275	85	24.8	27.8	1,200,000
Elm, rock	1,500	1,200	500	125	21.1	23.6	1,300,000
Elm, slippery and white	1,100	800	250	100	24.8	27.8	1,200,000
Fir, balsam	900	700	150	70	24.2	27.1	1,000,000
Fir, commercial white	1,100	700	300	70	25.4	28.4	1,100,000
Gum, red, black and cotton	1,100	800	300	100	24.8	27.8	1,200,000
Hemlock, western	1,300	900	300	75	25.3	28.3	1,400,000
Hemlock, eastern	1,100	700	300	70	25.4	28.4	1,100,000
Hickory (true and pecan)	1,900	1,500	600	140	22.2	24.8	1,800,000
Larch, western	1,200	1,100	325	100	22.0	24.6	1,300,000
Maple, sugar and black	1,500	1,200	500	125	23.4	26.2	1,600,000
Maple, red and silver	1,000	800	350	100	23.8	26.6	1,100,000
Oak, commercial red and white	1,400	1,000	500	125	24.8	27.8	1,500,000
Pine, southern yellow	1,600	1,175	345	110	23.7	27.3	1,600,000
Pine, southern yellow, dense	1,750	1,290	380	120	22.6	24.9	1,600,000
Pine, sugar, white, western white and western yellow	900	750	250	85	23.4	26.2	1,000,000
Pine, Norway	1,100	800	300	85	24.8	27.8	1,200,000
Poplar, yellow	1,000	800	250	80	23.8	26.8	1,100,000
Redwood	1,200	1,000	250	70	22.2	24.8	1,200,000
Spruce, red, white and Sitka	1,100	800	250	85	24.8	27.8	1,200,000
Spruce, Engelman	750	600	175	70	23.4	26.2	800,000
Sycamore	1,100	800	300	80	24.8	27.8	1,200,000
Tamarack, eastern	1,200	1,000	300	95	23.1	25.8	1,300,000

TABLE VI.—COEFFICIENT OF TEMPERATURE

Material	ϵ Fahrenheit	ϵ Centigrade
Cast iron	0.0000060	0.0000108
Wrought iron	0.0000067	0.0000121
Steel	0.0000065	0.0000117
Brass	0.0000105	0.0000190
Copper	0.0000093	0.0000168
Concrete	0.0000079	0.0000143
Glass	0.0000044	0.0000080

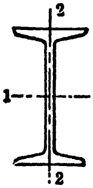
TABLE VII.—VALUES SPECIFIED BY THE AMERICAN SOCIETY FOR TESTING MATERIALS

Material	Ultimate strength, pounds per square inch	Yield point, pounds per square inch	Minimum elongation		Minimum reduction of area, per cent
			In 8 inches, per cent	In 2 inches, per cent	
Wrought iron	48,000	25,000	22		
Rivet steel for bridges	46– 56,000	½ ult. str.	$\frac{1,500,000}{\text{ult. str.}}$		
Structural steel for bridges	55– 65,000	½ ult. str.	$\frac{1,500,000}{\text{ult. str.}}$	22	
Structural nickel steel	85–100,000	50,000	$\frac{1,500,000}{\text{ult. str.}}$	18	25
Reinforcing bars (deformed)	80,000 (min.)	50,000	$\frac{1,000,000}{\text{ult. str.}}$		

TABLE VIII.—SHEARING MODULUS OF ELASTICITY

Material	E_s Pounds per square inch
Cast iron	6,000,000
Wrought iron	10,000,000
Steel	12,000,000
Aluminum	4,000,000
Brass	4,800,000
Copper	6,400,000

TABLE IX.[1]—PROPERTIES OF AMERICAN STANDARD BEAMS



Depth of beam	Weight per foot	Area of section	Width of flange	Thickness of web	Axis 1-1			Axis 2-2		
					I	k	$\frac{I}{c}$	I	k	$\frac{I}{c}$
In.	Lb.	In.^{2}	In.	In.	In.^{4}	In.	In.^{3}	In.^{4}	In.	In.^{3}
24	120.0	35.13	8.048	0.798	3,010.8	9.26	250.9	84.9	1.56	21.1
	115.0	33.67	7.987	0.737	2,940.5	9.35	245.0	82.8	1.57	20.7
	110.0	32.18	7.925	0.675	2,869.1	9.44	239.1	80.6	1.58	20.3
	105.9	30.98	7.875	0.625	2,811.5	9.53	234.3	78.9	1.60	20.0
24	100.0	29.25	7.247	0.747	2,371.8	9.05	197.6	48.4	1.29	13.4
	95.0	27.79	7.186	0.686	2,301.5	9.08	191.8	47.0	1.30	13.0
	90.0	26.30	7.124	0.624	2,230.1	9.21	185.8	45.5	1.32	12.8
	85.0	24.84	7.063	0.563	2,159.8	9.33	180.0	44.2	1.33	12.5
	79.9	23.33	7.000	0.500	2,087.2	9.46	173.9	42.9	1.36	12.2
20	100.0	29.20	7.273	0.873	1,648.3	7.51	164.8	52.4	1.34	14.4
	95.0	27.74	7.200	0.800	1,599.7	7.59	160.0	50.5	1.35	14.0
	90.0	26.26	7.126	0.726	1,550.3	7.68	155.0	48.7	1.36	13.7
	85.0	24.80	7.053	0.653	1,501.7	7.78	150.2	47.0	1.38	13.3
	81.4	23.74	7.000	0.600	1,466.3	7.86	146.6	45.8	1.39	13.1
20	75.0	21.90	6.391	0.641	1,263.5	7.60	126.3	30.1	1.17	9.4
	70.0	20.42	6.317	0.567	1,214.2	7.71	121.4	28.9	1.19	9.2
	65.4	19.08	6.250	0.500	1,169.5	7.83	116.9	27.9	1.21	8.9
18	90.0	26.29	7.236	0.796	1,256.5	6.91	139.6	51.9	1.40	14.3
	85.0	24.81	7.154	0.714	1,216.6	7.00	135.2	49.8	1.42	14.0
	80.0	23.34	7.072	0.632	1,176.8	7.10	130.8	47.9	1.43	13.6
	75.6	22.04	7.000	0.560	1,141.8	7.20	126.9	46.3	1.45	13.2
18	70.0	20.46	6.251	0.711	917.5	6.70	101.9	24.5	1.09	7.8
	65.0	18.98	6.169	0.629	877.7	6.80	97.5	23.4	1.11	7.6
	60.0	17.50	6.087	0.547	837.8	6.92	93.1	22.3	1.13	7.3
	54.7	15.94	6.000	0.460	795.5	7.07	83.4	21.2	1.15	7.1
15	75.0	21.85	6.278	0.868	687.2	5.61	91.6	30.6	1.18	9.8
	70.0	20.38	6.180	0.770	659.6	5.69	87.9	28.8	1.19	9.3
	65.0	18.91	6.082	0.672	632.1	5.78	84.3	27.2	1.20	8.9
	60.8	17.68	6.000	0.590	609.0	5.87	81.2	26.0	1.21	8.7
15	55.0	16.06	5.738	0.648	508.7	5.63	67.8	17.0	1.03	5.9
	50.0	14.59	5.640	0.550	481.1	5.74	64.2	16.0	1.05	5.7
	45.0	13.12	5.542	0.452	453.6	5.88	60.5	15.0	1.07	5.4
	42.9	12.49	5.500	0.410	441.8	5.95	58.9	14.6	1.08	5.3

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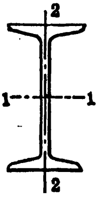
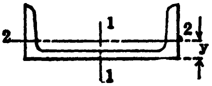
[1] Tables IX–XII are reprinted, with slight changes, from the Carnegie Pocket Companion, by courtesy of the Carnegie Steel Company.

TABLE IX.—PROPERTIES OF AMERICAN STANDARD BEAMS.—*Continued*



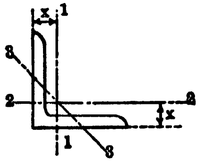
Depth of beam	Weight per foot	Area of section	Width of flange	Thickness of web	Axis 1-1			Axis 2-2		
					I	k	$\frac{I}{c}$	I	k	$\frac{I}{c}$
In.	Lb.	In.^{2}	In.	In.	In.^{4}	In.	In.^{3}	In.^{4}	In.	In.^{3}
12	55.0	16.04	5.600	0.810	319.3	4.46	53.2	17.3	1.04	6.2
	50.0	14.57	5.477	0.687	301.6	4.55	50.3	16.0	1.05	5.8
	45.0	13.10	5.355	0.565	284.1	4.66	47.3	14.8	1.06	5.5
	40.8	11.84	5.250	0.460	268.9	4.77	44.8	13.8	1.08	5.3
12	35.0	10.20	5.078	0.428	227.0	4.72	37.8	10.0	0.99	3.9
	31.8	9.26	5.000	0.350	215.8	4.83	36.0	9.5	1.01	3.8
10	40.0	11.69	5.091	0.741	158.0	3.68	31.6	9.4	0.90	3.7
	35.0	10.22	4.944	0.594	145.8	3.78	29.2	8.5	0.91	3.4
	30.0	8.75	4.797	0.447	133.5	3.91	26.7	7.6	0.93	3.2
	25.4	7.38	4.660	0.310	122.1	4.07	24.4	6.9	0.97	3.0
9	35.0	10.22	4.764	0.724	111.3	3.30	24.7	7.3	0.84	3.0
	30.0	8.76	4.601	0.561	101.4	3.40	22.5	6.4	0.85	2.8
	25.0	7.28	4.437	0.397	91.4	3.54	20.3	5.6	0.88	2.5
	21.8	6.32	4.330	0.290	84.9	3.67	18.9	5.2	0.90	2.4
8	25.5	7.43	4.262	0.532	68.1	3.03	17.0	4.7	0.80	2.2
	23.0	6.71	4.171	0.441	64.2	3.09	16.0	4.4	0.81	2.1
	20.5	5.97	4.079	0.349	60.2	3.18	15.1	4.0	0.82	2.0
	18.4	5.34	4.000	0.270	56.9	3.26	14.2	3.8	0.84	1.9
7	20.0	5.83	3.860	0.450	41.9	2.68	12.0	3.1	0.74	1.6
	17.5	5.09	3.755	0.345	38.9	2.77	11.1	2.9	0.76	1.6
	15.3	4.43	3.660	0.250	36.2	2.86	10.4	2.7	0.78	1.5
6	17.25	5.02	3.565	0.465	26.0	2.28	8.7	2.3	0.68	1.3
	14.75	4.29	3.443	0.343	23.8	2.36	7.9	2.1	0.69	1.2
	12.5	3.61	3.330	0.230	21.8	2.46	7.3	1.8	0.72	1.1
5	14.75	4.29	3.284	0.494	15.0	1.87	6.0	1.7	0.63	1.0
	12.25	3.56	3.137	0.347	13.5	1.95	5.4	1.4	0.63	0.91
	10.0	2.87	3.000	0.210	12.1	2.05	4.8	1.2	0.65	0.82
4	10.5	3.05	2.870	0.400	7.1	1.52	3.5	1.0	0.57	0.70
	9.5	2.76	2.796	0.326	6.7	1.56	3.3	0.91	0.58	0.65
	8.5	2.46	2.723	0.253	6.3	1.60	3.2	0.83	0.58	0.61
	7.7	2.21	2.660	0.190	6.0	1.64	3.0	0.77	0.59	0.58
3	7.5	2.17	2.509	0.349	2.9	1.15	1.9	0.59	0.52	0.47
	6.5	1.88	2.411	0.251	2.7	1.19	1.8	0.51	0.52	0.43
	5.7	1.64	2.330	0.170	2.5	1.23	1.7	0.46	0.53	0.40

TABLE X.—PROPERTIES OF STRUCTURAL CHANNELS



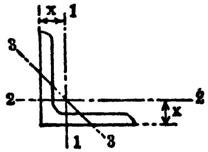
Depth of channel	Weight per foot	Area of section	Width of flange	Thickness of web	Axis 1-1			Axis 2-2			
					I	k	$\frac{I}{c}$	I	k	$\frac{I}{c}$	y
In.	Lb.	In.^{2}	In.	In.	In.^{4}	In.	In.^{3}	In.^{4}	In.	In.^{3}	In.
15	55.0	16.11	3.814	0.814	429.0	5.16	57.2	12.1	0.87	4.1	0.82
	50.0	14.64	3.716	0.716	401.4	5.24	53.6	11.2	0.87	3.8	0.80
	45.0	13.17	3.618	0.618	373.9	5.33	49.8	10.3	0.88	3.6	0.79
	40.0	11.70	3.520	0.520	346.3	5.44	46.2	9.3	0.89	3.4	0.78
	35.0	10.23	3.422	0.422	318.7	5.58	42.5	8.4	0.91	3.2	0.79
	33.9	9.90	3.400	0.400	312.6	5.62	41.7	8.2	0.91	3.2	0.79
12	40.0	11.73	3.415	0.755	196.5	4.09	32.8	6.6	0.75	2.5	0.72
	35.0	10.26	3.292	0.632	178.8	4.18	29.8	5.9	0.76	2.3	0.69
	30.0	8.79	3.170	0.510	161.2	4.28	26.9	5.2	0.77	2.1	0.68
	25.0	7.32	3.047	0.387	143.5	4.43	23.9	4.5	0.79	1.9	0.68
	20.7	6.03	2.940	0.280	128.1	4.61	21.4	3.9	0.81	1.7	0.70
10	35.0	10.27	3.180	0.820	115.2	3.34	23.0	4.6	0.67	1.9	0.69
	30.0	8.80	3.033	0.673	103.0	3.42	20.6	4.0	0.67	1.7	0.65
	25.0	7.33	2.886	0.526	90.7	3.52	18.1	3.4	0.68	1.5	0.62
	20.0	5.86	2.739	0.379	78.5	3.66	15.7	2.8	0.70	1.3	0.61
	15.3	4.47	2.600	0.240	66.9	3.87	13.4	2.3	0.72	1.2	0.64
9	25.0	7.33	2.812	0.612	70.5	3.10	15.7	3.0	0.64	1.4	0.61
	20.0	5.86	2.648	0.448	60.6	3.22	13.5	2.4	0.65	1.2	0.59
	15.0	4.39	2.485	0.285	50.7	3.40	11.3	1.9	0.67	1.0	0.59
	13.4	3.89	2.430	0.230	47.3	3.49	10.5	1.8	0.67	0.97	0.61
8	21.25	6.23	2.619	0.579	47.6	2.77	11.9	2.2	0.60	1.1	0.59
	18.75	5.49	2.527	0.487	43.7	2.82	10.9	2.0	0.60	1.0	0.57
	16.25	4.76	2.435	0.395	39.8	2.89	9.9	1.8	0.61	0.94	0.56
	13.75	4.02	2.343	0.303	35.8	2.99	9.0	1.5	0.62	0.86	0.56
	11.5	3.36	2.260	0.220	32.3	3.10	8.1	1.3	0.63	0.79	0.58
7	19.75	5.79	2.509	0.629	33.1	2.39	9.4	1.8	0.56	0.96	0.58
	17.25	5.05	2.404	0.524	30.1	2.44	8.6	1.6	0.56	0.86	0.55
	14.75	4.32	2.299	0.419	27.1	2.51	7.7	1.4	0.57	0.79	0.53
	12.25	3.58	2.194	0.314	24.1	2.59	6.9	1.2	0.58	0.71	0.53
	9.8	2.85	2.090	0.210	21.1	2.72	6.0	0.98	0.59	0.63	0.55
6	15.5	4.54	2.279	0.559	19.5	2.07	6.5	1.3	0.53	0.73	0.55
	13.0	3.81	2.157	0.437	17.3	2.13	5.8	1.1	0.53	0.65	0.52
	10.5	3.07	2.034	0.314	15.1	2.22	5.0	0.87	0.53	0.57	0.50
	8.2	2.39	1.920	0.200	13.0	2.34	4.3	0.70	0.54	0.50	0.52
5	11.5	3.36	2.032	0.472	10.4	1.76	4.1	0.82	0.49	0.54	0.51
	9.0	2.63	1.885	0.325	8.8	1.83	3.5	0.64	0.49	0.45	0.48
	6.7	1.95	1.750	0.190	7.4	1.95	3.0	0.48	0.50	0.38	0.49
4	7.25	2.12	1.720	0.320	4.5	1.47	2.3	0.44	0.46	0.35	0.46
	6.25	1.82	1.647	0.247	4.1	1.50	2.1	0.38	0.45	0.32	0.46
	5.4	1.56	1.580	0.180	3.8	1.56	1.9	0.32	0.45	0.29	0.46
3	6.0	1.75	1.596	0.356	2.1	1.08	1.4	0.31	0.42	0.27	0.46
	5.0	1.46	1.498	0.258	1.8	1.12	1.2	0.25	0.41	0.24	0.44
	4.1	1.19	1.410	0.170	1.6	1.17	1.1	0.20	0.41	0.21	0.44

TABLE XI.—PROPERTIES OF EQUAL ANGLES



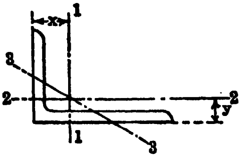
Size	Thickness	Weight per foot	Area of section	Axis 1-1 and Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I min.	k min.
In.	In.	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
8 × 8	1 1/8	56.9	16.73	98.0	2.42	17.5	2.41	40.2	1.55
	1 1/16	54.0	15.87	93.5	2.43	16.7	2.39	38.6	1.56
	1	51.0	15.00	89.0	2.44	15.8	2.37	36.5	1.56
	15/16	48.1	14.12	84.3	2.44	14.9	2.34	34.3	1.56
	7/8	45.0	13.23	79.6	2.45	14.0	2.32	32.2	1.56
	13/16	42.0	12.34	74.7	2.46	13.1	2.30	30.4	1.57
	3/4	38.9	11.44	69.7	2.47	12.2	2.28	28.2	1.57
	11/16	35.8	10.53	64.6	2.48	11.2	2.25	26.2	1.58
	5/8	32.7	9.61	59.4	2.49	10.3	2.23	23.9	1.58
	9/16	29.6	8.68	54.1	2.50	9.3	2.21	21.6	1.58
	1/2	26.4	7.75	48.6	2.51	8.4	2.19	19.3	1.58
6 × 6	1	37.4	11.00	35.5	1.80	8.6	1.86	14.8	1.16
	15/16	35.3	10.37	33.7	1.80	8.1	1.84	13.9	1.16
	7/8	33.1	9.73	31.9	1.81	7.6	1.82	13.3	1.17
	13/16	31.0	9.09	30.1	1.82	7.2	1.80	12.4	1.17
	3/4	28.7	8.44	28.2	1.83	6.7	1.78	11.5	1.17
	11/16	26.5	7.78	26.2	1.83	6.2	1.75	10.6	1.17
	5/8	24.2	7.11	24.2	1.84	5.7	1.73	9.7	1.17
	9/16	21.9	6.43	22.1	1.85	5.1	1.71	9.0	1.18
	1/2	19.6	5.75	19.9	1.86	4.6	1.68	8.0	1.18
	7/16	17.2	5.06	17.7	1.87	4.1	1.66	7.2	1.19
	3/8	14.9	4.36	15.4	1.88	3.5	1.64	6.2	1.19
5 × 5	1	30.6	9.00	19.6	1.48	5.8	1.61	8.3	0.96
	15/16	28.9	8.50	18.7	1.48	5.5	1.59	7.8	0.96
	7/8	27.2	7.98	17.8	1.49	5.2	1.57	7.3	0.96
	13/16	25.4	7.47	16.8	1.50	4.9	1.55	7.0	0.97
	3/4	23.6	6.94	15.7	1.50	4.5	1.52	6.5	0.97
	11/16	21.8	6.40	14.7	1.51	4.2	1.50	6.0	0.97
	5/8	20.0	5.86	13.6	1.52	3.9	1.48	5.5	0.97
	9/16	18.1	5.31	12.4	1.53	3.5	1.46	5.1	0.98
	1/2	16.2	4.75	11.3	1.54	3.2	1.43	4.6	0.98
	7/16	14.3	4.18	10.0	1.55	2.8	1.41	4.0	0.98
	3/8	12.3	3.61	8.7	1.56	2.4	1.39	3.5	0.99
4 × 4	13/16	19.9	5.84	8.1	1.18	3.0	1.29	3.5	0.77
	3/4	18.5	5.44	7.7	1.19	2.8	1.27	3.2	0.77
	11/16	17.1	5.03	7.2	1.19	2.6	1.25	3.0	0.77
	5/8	15.7	4.61	6.7	1.20	2.4	1.23	2.7	0.77
	9/16	14.3	4.18	6.1	1.21	2.2	1.21	2.5	0.78
	1/2	12.8	3.75	5.6	1.22	2.0	1.18	2.3	0.78
	7/16	11.3	3.31	5.0	1.23	1.8	1.16	2.0	0.78
	3/8	9.8	2.86	4.4	1.23	1.5	1.14	1.8	0.79
	5/16	8.2	2.40	3.7	1.24	1.3	1.12	1.5	0.79
	1/4	6.6	1.94	3.0	1.25	1.0	1.09	1.2	0.79

TABLE XI.—PROPERTIES OF EQUAL ANGLES.—*Continued*



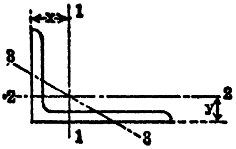
Size	Thickness	Weight per foot	Area of section	Axis 1-1 and Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I min.	k min.
In.	In.	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
3½ × 3½	13/16	17.1	5.03	5.3	1.02	2.3	1.17	2.3	0.67
	¾	16.0	4.69	5.0	1.03	2.1	1.15	2.1	0.67
	11/16	14.8	4.34	4.7	1.04	2.0	1.12	1.9	0.67
	⅝	13.6	3.98	4.3	1.04	1.8	1.10	1.8	0.68
	9/16	12.4	3.62	4.0	1.05	1.6	1.08	1.7	0.68
	½	11.1	3.25	3.6	1.06	1.5	1.06	1.5	0.68
	7/16	9.8	2.87	3.3	1.07	1.3	1.04	1.3	0.68
	⅜	8.5	2.48	2.9	1.07	1.2	1.01	1.2	0.69
	5/16	7.2	2.09	2.5	1.08	0.98	0.99	1.0	0.69
	¼	5.8	1.69	2.0	1.09	0.79	0.97	0.80	0.69
3 × 3	⅝	11.5	3.36	2.6	0.88	1.3	0.98	1.1	0.57
	9/16	10.4	3.06	2.4	0.89	1.2	0.95	1.0	0.58
	½	9.4	2.75	2.2	0.90	1.1	0.93	0.93	0.58
	7/16	8.3	2.43	2.0	0.91	0.95	0.91	0.82	0.58
	⅜	7.2	2.11	1.8	0.91	0.83	0.89	0.71	0.58
	5/16	6.1	1.78	1.5	0.92	0.71	0.87	0.62	0.59
	¼	4.9	1.44	1.2	0.93	0.58	0.84	0.50	0.59
2½ × 2½	½	7.7	2.25	1.2	0.74	0.73	0.81	0.50	0.47
	7/16	6.8	2.00	1.1	0.75	0.65	0.78	0.46	0.48
	⅜	5.9	1.73	0.98	0.75	0.57	0.76	0.40	0.48
	5/16	5.0	1.47	0.85	0.76	0.48	0.74	0.35	0.49
	¼	4.1	1.19	0.70	0.77	0.39	0.72	0.29	0.49
	3/16	3.07	0.90	0.55	0.78	0.30	0.69	0.22	0.49
	⅛	2.08	0.61	0.38	0.79	0.20	0.67	0.15	0.50
2 × 2	7/16	5.3	1.56	0.54	0.59	0.40	0.66	0.24	0.39
	⅜	4.7	1.36	0.48	0.59	0.35	0.64	0.21	0.39
	5/16	3.92	1.15	0.42	0.60	0.30	0.61	0.18	0.39
	¼	3.19	0.94	0.35	0.61	0.25	0.59	0.14	0.39
	3/16	2.44	0.71	0.28	0.62	0.19	0.57	0.11	0.40
	⅛	1.65	0.48	0.19	0.63	0.13	0.55	0.08	0.40
1¾ × 1¾	7/16	4.6	1.34	0.35	0.51	0.30	0.59	0.15	0.33
	⅜	3.99	1.17	0.31	0.51	0.26	0.57	0.14	0.34
	5/16	3.39	1.00	0.27	0.52	0.23	0.55	0.12	0.34
	¼	2.77	0.81	0.23	0.53	0.19	0.53	0.09	0.34
	3/16	2.12	0.62	0.18	0.54	0.14	0.51	0.08	0.35
	⅛	1.44	0.42	0.13	0.55	0.10	0.48	0.05	0.35
1½ × 1½	⅜	3.35	0.98	0.19	0.44	0.19	0.51	0.08	0.29
	5/16	2.86	0.84	0.16	0.44	0.16	0.49	0.07	0.29
	¼	2.34	0.69	0.14	0.45	0.13	0.47	0.06	0.29
	3/16	1.80	0.53	0.11	0.46	0.10	0.44	0.04	0.29
	⅛	1.23	0.36	0.08	0.46	0.07	0.42	0.03	0.30
1¼ × 1¼	5/16	2.33	0.68	0.09	0.36	0.11	0.42	0.04	0.24
	¼	1.92	0.56	0.08	0.37	0.09	0.40	0.03	0.24
	3/16	1.48	0.43	0.06	0.38	0.07	0.38	0.02	0.24
	⅛	1.01	0.30	0.04	0.38	0.05	0.35	0.02	0.25
1 × 1	¼	1.49	0.44	0.04	0.29	0.06	0.34	0.02	0.19
	3/16	1.16	0.34	0.03	0.30	0.04	0.32	0.01	0.19
	⅛	0.80	0.23	0.02	0.31	0.03	0.30	0.01	0.19

TABLE XII.—PROPERTIES OF UNEQUAL ANGLES



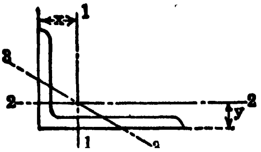
Size	Thickness	Weight per foot	Area of section	Axis 1-1				Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I	k	$\frac{I}{c}$	y	I min.	k min.
In.	In.	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
8 × 6	1	44.2	13.00	80.8	2.49	15.1	2.65	38.8	1.73	8.9	1.65	21.3	1.28
	$\frac{15}{16}$	41.7	12.25	76.6	2.50	14.3	2.63	36.8	1.73	8.4	1.63	20.1	1.28
	$\frac{7}{8}$	39.1	11.48	72.3	2.51	13.4	2.61	34.9	1.74	7.9	1.61	18.8	1.28
	$\frac{13}{16}$	36.5	10.72	67.9	2.52	12.5	2.59	32.8	1.75	7.4	1.59	17.8	1.29
	$\frac{3}{4}$	33.8	9.94	63.4	2.53	11.7	2.56	30.7	1.76	6.9	1.56	16.5	1.29
	$\frac{11}{16}$	31.2	9.15	58.8	2.54	10.8	2.54	28.6	1.77	6.4	1.54	15.2	1.29
	$\frac{5}{8}$	28.5	8.36	54.1	2.54	9.9	2.52	26.3	1.77	5.9	1.52	14.1	1.30
	$\frac{9}{16}$	25.7	7.56	49.3	2.55	8.9	2.50	24.0	1.78	5.3	1.50	12.8	1.30
	$\frac{1}{2}$	23.0	6.75	44.3	2.56	8.0	2.47	21.7	1.79	4.8	1.47	11.4	1.30
	$\frac{7}{16}$	20.2	5.93	39.2	2.57	7.1	2.45	19.3	1.80	4.2	1.45	10.0	1.30
8 × 3½	1	35.7	10.50	66.2	2.51	13.7	3.17	7.8	0.86	3.0	0.92	5.6	0.73
	$\frac{15}{16}$	33.7	9.90	62.9	2.52	12.9	3.14	7.4	0.87	2.9	0.89	5.3	0.73
	$\frac{7}{8}$	31.7	9.30	59.4	2.53	12.2	3.12	7.1	0.87	2.7	0.87	5.0	0.73
	$\frac{13}{16}$	29.6	8.68	55.9	2.54	11.4	3.10	6.7	0.88	2.5	0.85	4.6	0.73
	$\frac{3}{4}$	27.5	8.00	52.3	2.55	10.6	3.07	6.3	0.88	2.3	0.82	4.3	0.73
	$\frac{11}{16}$	25.3	7.43	48.5	2.56	9.8	3.05	5.9	0.89	2.2	0.80	4.0	0.73
	$\frac{5}{8}$	23.2	6.80	44.7	2.57	9.0	3.03	5.4	0.90	2.0	0.78	3.7	0.74
	$\frac{9}{16}$	21.0	6.15	40.8	2.57	8.2	3.00	5.0	0.90	1.8	0.75	3.4	0.74
	$\frac{1}{2}$	18.7	5.50	36.7	2.58	7.3	2.98	4.5	0.91	1.6	0.73	3.0	0.74
	$\frac{7}{16}$	16.5	4.84	32.5	2.59	6.4	2.95	4.1	0.92	1.5	0.70	2.6	0.74
7 × 3½	1	32.3	9.50	45.4	2.19	10.6	2.70	7.5	0.89	3.0	0.96	5.2	0.74
	$\frac{15}{16}$	30.5	8.97	43.1	2.19	10.0	2.69	7.2	0.89	2.8	0.94	4.9	0.74
	$\frac{7}{8}$	28.7	8.42	40.8	2.20	9.4	2.66	6.8	0.90	2.6	0.91	4.6	0.74
	$\frac{13}{16}$	26.8	7.87	38.4	2.21	8.8	2.64	6.5	0.91	2.5	0.89	4.3	0.74
	$\frac{3}{4}$	24.9	7.31	36.0	2.22	8.2	2.62	6.1	0.91	2.3	0.87	4.0	0.74
	$\frac{11}{16}$	23.0	6.75	33.5	2.23	7.6	2.60	5.7	0.92	2.1	0.85	3.7	0.74
	$\frac{5}{8}$	21.0	6.17	30.9	2.24	7.0	2.57	5.3	0.93	2.0	0.82	3.5	0.75
	$\frac{9}{16}$	19.1	5.59	28.2	2.25	6.3	2.55	4.9	0.93	1.8	0.80	3.1	0.75
	$\frac{1}{2}$	17.0	5.00	25.4	2.25	5.7	2.53	4.4	0.94	1.6	0.78	2.8	0.75
	$\frac{7}{16}$	15.0	4.40	22.6	2.26	5.0	2.50	4.0	0.95	1.4	0.75	2.5	0.76
	$\frac{3}{8}$	13.0	3.80	19.6	2.27	4.3	2.48	3.5	0.96	1.3	0.73	2.2	0.76
6 × 4	1	30.6	9.00	30.8	1.85	8.0	2.17	10.8	1.09	3.8	1.17	6.5	0.85
	$\frac{15}{16}$	28.9	8.50	29.3	1.86	7.6	2.14	10.3	1.10	3.6	1.14	6.1	0.85
	$\frac{7}{8}$	27.2	7.98	27.7	1.86	7.2	2.12	9.8	1.11	3.4	1.12	5.9	0.86
	$\frac{13}{16}$	25.4	7.47	26.1	1.87	6.7	2.10	9.2	1.11	3.2	1.10	5.5	0.86
	$\frac{3}{4}$	23.6	6.94	24.5	1.88	6.2	2.08	8.7	1.12	3.0	1.08	5.1	0.86
	$\frac{11}{16}$	21.8	6.40	22.8	1.89	5.8	2.06	8.1	1.13	2.8	1.06	4.7	0.86
	$\frac{5}{8}$	20.0	5.86	21.1	1.90	5.3	2.03	7.5	1.13	2.5	1.03	4.3	0.86
	$\frac{9}{16}$	18.1	5.31	19.3	1.90	4.8	2.01	6.9	1.14	2.3	1.01	4.0	0.87
	$\frac{1}{2}$	16.2	4.75	17.4	1.91	4.3	1.99	6.3	1.15	2.1	0.99	3.6	0.87
	$\frac{7}{16}$	14.3	4.18	15.5	1.92	3.8	1.96	5.6	1.16	1.8	0.96	3.2	0.87
	$\frac{3}{8}$	12.3	3.61	13.5	1.93	3.3	1.94	4.9	1.17	1.6	0.94	2.8	0.88

TABLE XII.—PROPERTIES OF UNEQUAL ANGLES.—*Continued*



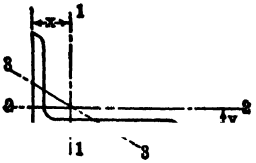
Size	Thickness	Weight per foot	Area of section	Axis 1-1				Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I	k	$\frac{I}{c}$	y	I min.	k min.
In.	In.	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
6 × 3½	1	28.9	8.50	29.2	1.85	7.8	2.26	7.2	0.92	2.9	1.01	4.7	0.74
	15/16	27.3	8.03	27.8	1.86	7.4	2.24	6.9	0.93	2.7	0.99	4.4	0.74
	7/8	25.7	7.55	26.4	1.87	7.0	2.22	6.6	0.93	2.6	0.97	4.2	0.75
	13/16	24.0	7.06	24.9	1.88	6.6	2.20	6.2	0.94	2.4	0.95	4.0	0.75
	3/4	22.4	6.56	23.3	1.89	6.1	2.18	5.8	0.94	2.3	0.93	3.7	0.75
	11/16	20.6	6.06	21.7	1.89	5.6	2.15	5.5	0.95	2.1	0.90	3.4	0.75
	5/8	18.9	5.55	20.1	1.90	5.2	2.13	5.1	0.96	1.9	0.88	3.1	0.75
	9/16	17.1	5.03	18.4	1.91	4.7	2.11	4.7	0.96	1.8	0.86	2.8	0.75
	1/2	15.3	4.50	16.6	1.92	4.2	2.08	4.3	0.97	1.6	0.83	2.6	0.76
	7/16	13.5	3.97	14.8	1.93	3.7	2.06	3.8	0.98	1.4	0.81	2.3	0.76
	3/8	11.7	3.42	12.9	1.94	3.3	2.04	3.3	0.99	1.2	0.78	2.0	0.77
	5/16	9.8	2.87	10.9	1.95	2.7	2.02	2.9	1.00	1.0	0.75	1.7	0.77
5 × 4	7/8	24.2	7.11	16.4	1.52	5.0	1.71	9.2	1.14	3.3	1.21	5.0	0.84
	13/16	22.7	6.65	15.5	1.53	4.7	1.68	8.7	1.15	3.1	1.18	4.7	0.84
	3/4	21.1	6.19	14.6	1.54	4.4	1.66	8.2	1.15	2.9	1.16	4.4	0.84
	11/16	19.5	5.72	13.6	1.54	4.1	1.64	7.7	1.16	2.7	1.14	4.0	0.84
	5/8	17.8	5.23	12.6	1.55	3.7	1.62	7.1	1.17	2.5	1.12	3.7	0.84
	9/16	16.2	4.75	11.6	1.56	3.4	1.60	6.6	1.18	2.3	1.10	3.4	0.85
	1/2	14.5	4.25	10.5	1.57	3.1	1.57	6.0	1.18	2.0	1.07	3.1	0.85
	7/16	12.8	3.75	9.3	1.58	2.7	1.55	5.3	1.19	1.8	1.05	2.7	0.85
	3/8	11.0	3.23	8.1	1.59	2.3	1.53	4.7	1.20	1.6	1.03	2.4	0.86
5 × 3½	7/8	22.7	6.67	15.7	1.53	4.9	1.79	6.2	0.96	2.5	1.04	3.8	0.75
	13/16	21.3	6.25	14.8	1.54	4.6	1.77	5.9	0.97	2.4	1.02	3.5	0.75
	3/4	19.8	5.81	13.9	1.55	4.3	1.75	5.6	0.98	2.2	1.00	3.3	0.75
	11/16	18.3	5.37	13.0	1.56	4.0	1.72	5.2	0.98	2.1	0.97	3.0	0.75
	5/8	16.8	4.92	12.0	1.56	3.7	1.70	4.8	0.99	1.9	0.95	2.8	0.75
	9/16	15.2	4.47	11.0	1.57	3.3	1.68	4.4	1.00	1.7	0.93	2.5	0.75
	1/2	13.6	4.00	10.0	1.58	3.0	1.66	4.0	1.01	1.6	0.91	2.2	0.75
	7/16	12.0	3.53	8.9	1.59	2.6	1.63	3.6	1.01	1.4	0.88	2.0	0.76
	3/8	10.4	3.05	7.8	1.60	2.3	1.61	3.2	1.02	1.2	0.86	1.8	0.76
	5/16	8.7	2.56	6.6	1.61	1.9	1.59	2.7	1.03	1.0	0.84	1.5	0.76
5 × 3	13/16	19.9	5.84	14.0	1.55	4.5	1.86	3.7	0.80	1.7	0.86	2.4	0.64
	3/4	18.5	5.44	13.2	1.55	4.2	1.84	3.5	0.80	1.6	0.84	2.2	0.64
	11/16	17.1	5.03	12.3	1.56	3.9	1.82	3.3	0.81	1.5	0.82	2.1	0.64
	5/8	15.7	4.61	11.4	1.57	3.5	1.80	3.1	0.81	1.4	0.80	1.9	0.64
	9/16	14.3	4.18	10.4	1.58	3.2	1.77	2.8	0.82	1.3	0.77	1.8	0.65
	1/2	12.8	3.75	9.5	1.59	2.9	1.75	2.6	0.83	1.1	0.75	1.6	0.65
	7/16	11.3	3.31	8.4	1.60	2.6	1.73	2.3	0.84	1.0	0.73	1.4	0.65
	3/8	9.8	2.86	7.4	1.61	2.2	1.70	2.0	0.84	0.89	0.70	1.2	0.65
	5/16	8.2	2.40	6.3	1.61	1.9	1.68	1.8	0.85	0.75	0.68	1.0	0.66

TABLE XII.—PROPERTIES OF UNEQUAL ANGLES.—*Continued*



Size	Thickness	Weight per foot	Area of section	Axis 1-1				Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I	k	$\frac{I}{c}$	y	I min.	k min.
In.	In.	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
4½ × 3	13/16	18.5	5.43	10.3	1.38	3.6	1.65	3.6	0.81	1.7	0.90	2.2	0.64
	3/4	17.3	5.06	9.7	1.39	3.4	1.63	3.4	0.82	1.6	0.88	2.1	0.64
	11/16	16.0	4.68	9.1	1.39	3.1	1.60	3.2	0.83	1.5	0.85	1.9	0.64
	5/8	14.7	4.30	8.4	1.40	2.9	1.58	3.0	0.83	1.4	0.83	1.8	0.64
	9/16	13.3	3.90	7.8	1.41	2.6	1.56	2.8	0.85	1.3	0.81	1.6	0.64
	1/2	11.9	3.50	7.0	1.42	2.4	1.54	2.5	0.85	1.1	0.79	1.5	0.65
	7/16	10.6	3.09	6.3	1.43	2.1	1.51	2.3	0.85	1.0	0.76	1.3	0.65
	3/8	9.1	2.67	5.5	1.44	1.8	1.49	2.0	0.86	0.88	0.74	1.2	0.66
	5/16	7.7	2.25	4.7	1.44	1.5	1.47	1.7	0.87	0.75	0.72	0.98	0.66
4 × 3½	13/16	18.5	5.43	7.8	1.19	2.9	1.36	5.5	1.01	2.3	1.11	2.8	0.72
	3/4	17.3	5.06	7.3	1.20	2.8	1.34	5.2	1.01	2.1	1.09	2.6	0.72
	11/16	16.0	4.68	6.9	1.21	2.6	1.32	4.9	1.02	2.0	1.07	2.4	0.72
	5/8	14.7	4.30	6.4	1.22	2.4	1.29	4.5	1.03	1.8	1.04	2.2	0.72
	9/16	13.3	3.90	5.9	1.23	2.1	1.27	4.2	1.03	1.7	1.02	2.0	0.72
	1/2	11.9	3.50	5.3	1.23	1.9	1.25	3.8	1.04	1.5	1.00	1.8	0.72
	7/16	10.6	3.09	4.8	1.24	1.7	1.23	3.4	1.05	1.3	0.98	1.6	0.72
	3/8	9.1	2.67	4.2	1.25	1.5	1.21	3.0	1.06	1.2	0.96	1.4	0.73
	5/16	7.7	2.25	3.6	1.26	1.3	1.18	2.6	1.07	1.0	0.93	1.2	0.73
4 × 3	13/16	17.1	5.03	7.3	1.21	2.9	1.44	3.5	0.83	1.7	0.94	2.1	0.64
	3/4	16.0	4.69	6.9	1.22	2.7	1.42	3.3	0.84	1.6	0.92	1.9	0.64
	11/16	14.8	4.34	6.5	1.22	2.5	1.39	3.1	0.84	1.5	0.89	1.8	0.64
	5/8	13.6	3.98	6.0	1.23	2.3	1.37	2.9	0.85	1.4	0.87	1.6	0.64
	9/16	12.4	3.62	5.6	1.24	2.1	1.35	2.7	0.86	1.2	0.85	1.5	0.64
	1/2	11.1	3.25	5.0	1.25	1.9	1.33	2.4	0.86	1.1	0.83	1.3	0.64
	7/16	9.8	2.87	4.5	1.25	1.7	1.30	2.2	0.87	1.0	0.80	1.2	0.64
	3/8	8.5	2.48	4.0	1.26	1.5	1.28	1.9	0.88	0.87	0.78	1.0	0.64
	5/16	7.2	2.09	3.4	1.27	1.2	1.26	1.7	0.89	0.74	0.76	0.89	0.65
	1/4	5.8	1.69	2.8	1.28	1.0	1.24	1.4	0.89	0.60	0.74	0.72	0.65
3½ × 3	13/16	15.8	4.62	5.0	1.04	2.2	1.23	3.3	0.85	1.7	0.98	1.8	0.62
	3/4	14.7	4.31	4.7	1.04	2.1	1.21	3.1	0.85	1.5	0.96	1.7	0.62
	11/16	13.6	4.00	4.4	1.05	1.9	1.19	3.0	0.86	1.4	0.94	1.5	0.62
	5/8	12.5	3.67	4.1	1.06	1.8	1.17	2.8	0.87	1.3	0.92	1.4	0.62
	9/16	11.4	3.34	3.8	1.07	1.6	1.15	2.5	0.87	1.2	0.90	1.3	0.62
	1/2	10.2	3.00	3.5	1.07	1.5	1.13	2.3	0.88	1.1	0.88	1.2	0.62
	7/16	9.1	2.65	3.1	1.08	1.3	1.10	2.1	0.89	0.98	0.85	1.0	0.62
	3/8	7.9	2.30	2.7	1.09	1.1	1.08	1.8	0.90	0.85	0.83	0.89	0.62
	5/16	6.6	1.93	2.3	1.10	0.96	1.06	1.6	0.90	0.72	0.81	0.77	0.63
	1/4	5.4	1.56	1.9	1.11	0.78	1.04	1.3	0.91	0.58	0.79	0.62	0.63
3½ × 2½	11/16	12.5	3.65	4.1	1.06	1.9	1.27	1.7	0.69	0.99	0.77	1.0	0.53
	5/8	11.5	3.36	3.8	1.07	1.7	1.25	1.6	0.69	0.92	0.75	0.95	0.53
	9/16	10.4	3.06	3.6	1.08	1.6	1.23	1.5	0.70	0.84	0.73	0.86	0.53
	1/2	9.4	2.75	3.2	1.09	1.4	1.20	1.4	0.70	0.76	0.70	0.77	0.53
	7/16	8.3	2.43	2.9	1.09	1.3	1.18	1.2	0.71	0.68	0.68	0.71	0.54
	3/8	7.2	2.11	2.6	1.10	1.1	1.16	1.1	0.72	0.59	0.66	0.61	0.54
	5/16	6.1	1.78	2.2	1.11	0.93	1.14	0.94	0.73	0.50	0.64	0.52	0.54
	1/4	4.9	1.44	1.8	1.12	0.75	1.11	0.78	0.74	0.41	0.61	0.42	0.54

TABLE XII.—PROPERTIES OF UNEQUAL ANGLES.—*Continued*



Size	Thickness	Weight per foot	Area of section	Axis 1-1				Axis 2-2				Axis 3-3	
				I	k	$\frac{I}{c}$	x	I	k	$\frac{I}{c}$	y	I min.	k min.
In.	In.	Lb.	In.	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.
3 × 2½	9/16	9.5	2.78	2.3	0.91	1.2	1.02	1.4	0.72	0.82	0.77	0.75	0.52
	½	8.5	2.50	2.1	0.91	1.0	1.00	1.3	0.72	0.74	0.75	0.68	0.52
	7/16	7.6	2.21	1.9	0.92	0.93	0.98	1.2	0.73	0.66	0.73	0.60	0.52
	⅜	6.6	1.92	1.7	0.93	0.81	0.96	1.0	0.74	0.58	0.71	0.52	0.52
	5/16	5.6	1.62	1.4	0.94	0.69	0.93	0.90	0.74	0.49	0.68	0.46	0.53
	¼	4.5	1.31	1.2	0.95	0.56	0.91	0.74	0.75	0.40	0.66	0.37	0.53
3 × 2	½	7.7	2.25	1.9	0.92	1.0	1.08	0.67	0.55	0.47	0.58	0.42	0.43
	7/16	6.8	2.00	1.7	0.93	0.89	1.06	0.61	0.55	0.42	0.56	0.37	0.43
	⅜	5.9	1.73	1.5	0.94	0.78	1.04	0.54	0.56	0.37	0.54	0.32	0.43
	5/16	5.0	1.47	1.3	0.95	0.66	1.02	0.47	0.57	0.32	0.52	0.27	0.43
	¼	4.1	1.19	1.1	0.95	0.54	0.99	0.39	0.57	0.25	0.49	0.22	0.43
2½ × 2	½	6.8	2.00	1.1	0.75	0.70	0.88	0.64	0.56	0.46	0.63	0.35	0.42
	7/16	6.1	1.78	1.0	0.76	0.62	0.85	0.58	0.57	0.41	0.60	0.31	0.42
	⅜	5.3	1.55	0.91	0.77	0.55	0.83	0.51	0.58	0.36	0.58	0.27	0.42
	5/16	4.5	1.31	0.79	0.78	0.47	0.81	0.45	0.58	0.31	0.56	0.23	0.42
	¼	3.62	1.06	0.65	0.78	0.38	0.79	0.37	0.59	0.25	0.54	0.19	0.42
	3/16	2.75	0.81	0.51	0.79	0.29	0.76	0.29	0.60	0.20	0.51	0.15	0.43
	⅛	1.86	0.55	0.35	0.80	0.20	0.74	0.20	0.61	0.13	0.49	0.10	0.43
2½ × 1½	5/16	3.92	1.15	0.71	0.79	0.44	0.90	0.19	0.41	0.17	0.40	0.12	0.32
	¼	3.19	0.94	0.59	0.79	0.36	0.88	0.16	0.41	0.14	0.38	0.10	0.32
	3/16	2.44	0.72	0.46	0.80	0.28	0.85	0.13	0.42	0.11	0.35	0.08	0.33
2¼ × 1½	½	5.6	1.63	0.75	0.68	0.54	0.86	0.26	0.40	0.26	0.48	0.17	0.32
	7/16	5.0	1.45	0.68	0.69	0.48	0.83	0.24	0.41	0.23	0.46	0.15	0.32
	⅜	4.4	1.27	0.61	0.69	0.42	0.81	0.21	0.41	0.20	0.44	0.13	0.32
	5/16	3.66	1.07	0.53	0.70	0.36	0.79	0.19	0.42	0.17	0.42	0.11	0.32
	¼	2.98	0.88	0.44	0.71	0.30	0.77	0.16	0.42	0.14	0.39	0.09	0.32
	3/16	2.28	0.67	0.34	0.72	0.23	0.75	0.12	0.43	0.11	0.37	0.07	0.33
2 × 1½	⅜	3.99	1.17	0.43	0.61	0.34	0.71	0.21	0.42	0.20	0.46	0.12	0.32
	5/16	3.39	1.00	0.38	0.62	0.29	0.69	0.18	0.42	0.17	0.44	0.10	0.32
	¼	2.77	0.81	0.32	0.62	0.24	0.66	0.15	0.43	0.14	0.41	0.08	0.32
	3/16	2.12	0.62	0.25	0.63	0.18	0.64	0.12	0.44	0.11	0.39	0.06	0.32
	⅛	1.44	0.42	0.17	0.64	0.13	0.62	0.09	0.45	0.08	0.37	0.05	0.33
2 × 1¼	¼	2.55	0.75	0.30	0.63	0.23	0.71	0.09	0.34	0.10	0.33	0.05	0.27
	3/16	1.96	0.57	0.23	0.64	0.18	0.69	0.07	0.35	0.08	0.31	0.04	0.27
1¾ × 1¼	¼	2.34	0.69	0.20	0.54	0.18	0.60	0.09	0.35	0.10	0.35	0.05	0.27
	3/16	1.80	0.53	0.16	0.55	0.14	0.58	0.07	0.36	0.08	0.33	0.04	0.27
	⅛	1.23	0.36	0.11	0.56	0.00	0.56	0.05	0.37	0.05	0.31	0.03	0.27
1½ × 1¼	5/16	2.59	0.76	0.16	0.45	0.16	0.52	0.10	0.35	0.11	0.40	0.05	0.26
	¼	2.13	0.63	0.13	0.46	0.13	0.50	0.08	0.36	0.09	0.38	0.04	0.26
	3/16	1.64	0.48	0.10	0.46	0.10	0.48	0.07	0.37	0.07	0.35	0.03	0.26

Table XIII.—Average Weights and Strengths of American Hard or Deciduous Timber
(12 Per Cent Moisture Content)

Common name	Weight pounds per cubic foot	Bending, pounds per square inch			Compression parallel to grain, pounds per square inch		
		Elastic limit	Modulus of rupture	Modulus of elasticity	Elastic limit	Ultimate strength	Modulus of elasticity
Ash, white	41	9,800	15,600	1,780,000	5,950	7,500	1,890,000
Basswood	26	6,100	8,800	1,470,000	3,580	4,850	1,810,000
Beech	44	8,600	14,600	1,670,000	4,550	6,940	1,970,000
Birch, yellow	43	10,800	17,000	2,100,000	6,660	8,420	2,040,000
Cottonwood, eastern	29	5,900	8,700	1,390,000	3,550	5,100	1,400,000
Elm, American white	35	7,700	12,000	1,350,000	4,130	5,160	1,430,000
Elm, rock	44	8,100	15,000	1,550,000	4,780	7,200	2,030,000
Elm, slippery	37	7,900	13,200	1,500,000	4,670	6,470	1,690,000
Gum, blue	52	11,700	16,800	2,400,000	8,800	10,250	3,030,000
Gum, red	34	8,100	11,900	1,490,000	4,300	5,820	1,640,000
Hackberry	37	6,000	11,200	1,200,000	3,710	5,540	1,160,000
Hickory, shagbark	51	10,800	20,400	2,170,000	6,000	9,380	2,230,000
Locust, black	49	12,900	19,400	2,060,000	7,580	10,250	3,690,000
Maple, sugar	44	9,500	15,700	1,840,000	5,610	7,930	2,000,000
Oak, black	43	7,900	13,900	1,640,000	4,710	6,530	1,890,000
Oak, chestnut	46	9,200	13,500	1,590,000	4,810	6,930	2,170,000
Oak, live	62	8,800	18,400	1,970,000	5,600	8,800	3,940,000
Oak, post	47	7,600	13,300	1,510,000	3,880	6,620	2,000,000
Oak, red	44	8,500	14,300	1,820,000	4,430	6,820	2,050,000
Oak, swamp white	50	10,200	17,600	2,040,000	5,540	8,450	2,140,000
Oak, white	48	8,200	15,200	1,780,000	4,340	7,430	1,880,000
Poplar, yellow	28	6,200	9,300	1,520,000	3,530	5,420	2,140,000
Sycamore	35	6,500	10,100	1,430,000	3,780	5,480	1,660,000
Walnut, black	39	10,100	15,000	1,690,000	6,020	7,820	2,050,000
Average	42	8,640	14,420	1,700,000	5,020	7,150	2,080,000

TABLE XIV.—AVERAGE WEIGHTS AND STRENGTHS OF AMERICAN SOFT OR CONIFER TIMBER
(12 Per Cent Moisture Content)

Common name	Weight pounds per cubic foot	Bending, pounds per square inch			Compression parallel to grain, pounds per square inch		
		Elastic limit	Modulus of rupture	Modulus of elasticity	Elastic limit	Ultimate strength	Modulus of elasticity
Cedar, Alaska	29	6,700	10,100	1,270,000	5,330	5,800	1,400,000
Cedar, eastern red	33	3,700	8,700	870,000	5,000	5,900	1,100,000
Cedar, western red	23	5,200	7,700	1,170,000	4,400	5,180	1,470,000
Cedar, northern white	22	4,900	6,500	800,000	2,730	3,980	1,020,000
Cypress	32	7,200	10,600	1,440,000	5,270	6,590	1,740,000
Fir, Douglas, coast type	34	8,200	11,900	1,930,000	6,870	7,590	2,330,000
Fir, Douglas, mountain type	30	6,300	9,600	1,410,000	4,720	6,140	1,690,000
Fir, red	27	7,000	10,800	1,560,000	4,700	5,330	2,000,000
Hemlock, western	28	6,600	9,300	1,450,000	5,260	5,850	1,550,000
Pine, loblolly	38	9,200	13,000	1,930,000	5,980	8,200	2,070,000
Pine, lodgepole	29	6,900	9,500	1,350,000	4,630	5,540	1,580,000
Pine, longleaf	41	10,100	15,500	2,060,000	7,010	9,060	2,280,000
Pine, Norway	33	9,400	12,500	1,800,000	5,840	7,250	2,210,000
Pine, shortleaf	38	8,700	13,300	1,920,000	6,080	7,860	2,440,000
Pine, sugar	25	5,700	8,100	1,200,000	4,600	4,770	1,510,000
Pine, eastern white	25	6,000	8,900	1,290,000	3,870	4,910	1,560,000
Pine, western white	27	6,300	9,600	1,520,000	4,910	5,720	2,120,000
Pine, western yellow	28	6,400	9,300	1,260,000	4,260	5,340	1,390,000
Redwood	30	7,900	10,800	1,370,000	5,270	6,600	1,740,000
Spruce, Engelman	23	5,600	8,000	1,120,000	3,360	4,420	1,160,000
Spruce, red	28	6,800	10,200	1,530,000	4,960	5,910	1,860,000
Spruce, Sitka	27	6,500	9,700	1,540,000	4,110	5,260	1,480,000
Spruce, white	27	6,100	9,100	1,430,000	3,940	5,170	1,840,000
Tamarack	37	8,100	11,700	1,650,000	4,950	7,170	2,050,000
Average	29.7	7,300	10,200	1,450,000	4,920	6,060	1,730,000

TABLE XV.—SAFE STRESSES IN STRUCTURAL STEEL RECOMMENDED BY THE BUILDING CODE COMMITTEE, U. S. DEPARTMENT OF COMMERCE, 1926
(Pounds per Square Inch)

	Acceptable steel	Standard steel
Tension, rolled steel, on net section	16,000	18,000
Compression, rolled steel, maximum for short columns	12,500	14,000
Compression on rolled-steel columns, unit load	$16,000-60\frac{l}{k}$	$18,000-70\frac{l}{k}$
Flexure, stress in tension flange of rolled beams	16,000	18,000
Flexure, stress in compression flange, $\frac{l}{b}$ below 15	16,000	18,000
Flexure, stress in compression flange, $\frac{l}{b}$ between 15 and 40	$19,600-240\frac{l}{b}$	$22,000-270\frac{l}{b}$
Flexure, stress in pins	24,000	27,000
Bearing, plane faced or rolled surfaces	24,000	27,000
Shear, webs of girders, $\frac{d}{t}$ less than 43	10,700	12,000
Shear, webs of girders, $\frac{d}{t}$ greater than 43	$13,300-62\frac{d}{t}$	$15,000-70\frac{d}{t}$
Shear, pins and power-driven rivets	12,000	13,500
Shear, rough bolts and hand-driven rivets	9,000	10,000
Bearing, of plates on pins or power-driven rivets which are subjected to single shear	24,000	24,000
Bearing, of plates on pins or power-driven rivets which are subjected to double shear	30,000	30,000
Bearing, of plates on rough bolts or hand-driven rivets which are subjected to single shear	16,000	16,000
Bearing, of plates on rough bolts or hand-driven rivets which are subjected to double shear	20,000	20,000

In the table above, l is the unsupported length of the column or of the beam flange, k is the least radius of gyration of the column cross-section, b is the width of the beam flange, d is the unsupported height of the web and t is the thickness of the web of the girder.

TABLE XVI.—SAFE STRESSES IN STRUCTURAL STEEL
SPECIFICATIONS OF THE AMERICAN INSTITUTE OF STEEL CONSTRUCTION, 1941.
(Pounds per Square Inch)

	Standard steel
Tension, rolled steel, on net section	20,000
Tension, on rivets, computed on nominal diameter	15,000
Tension, on bolts, area at root of thread	12,000
Compression, axial, on columns, $\frac{l}{k}$ not over 120	$17{,}000 - 0.485\left(\frac{l}{k}\right)^2$
Compression, axial, on columns, $\frac{l}{k}$ above 120	$\dfrac{18{,}000}{1 + \frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}$
Flexure, stress in tension flange	20,000
Flexure, stress in compression flange, $\frac{l}{b}$ not over 40	$\dfrac{22{,}500}{1 + \frac{1}{1{,}800}\left(\frac{l}{b}\right)^2}$ with a maximum of 20,000
Flexure, on pins	30,000
Shear, webs of beams and plate girders	13,000
Shear, rivets, pins, and turned bolts in reamed or drilled holes	15,000
Shear, unfinished bolts	10,000
Bearing, rivets, and turned bolts in reamed or drilled holes, double shear	40,000
Bearing, rivets, and turned bolts in reamed or drilled holes, single shear	32,000
Bearing, on pins	32,000
Bearing, on unfinished bolts, double shear	25,000
Bearing, on unfinished bolts, single shear	20,000
Bearing, milled surfaces	30,000
Bearing, fitted stiffeners	27,000

In the table above, l is the unsupported length of the column or of the beam flange; k is the least radius of gyration of the column cross-section; and b is the width of the beam flange.

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